G. A. Wentworth.

SOLID GEOMETRY

BY

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PREFACE.

A SUFFICIENTLY full treatment of the theorems of limits is given in the Plane Geometry to lay the foundation for rigorous proofs of the theorems in Solid Geometry that involve limits. Theorems not found in previous editions are introduced in this edition wherever deemed necessary for satisfactory proofs of propositions that follow. In Book VII Theorem XVI, for example, is necessary for a logical proof of the important theorem that follows it, and Theorems XXII and XXVIII are likewise necessary for the proofs that follow on Cylinders and Cones, respectively.

While rigorous proofs have claimed first attention, still every effort has been made to bring the proofs within the comprehension of students who read the subject for the first time. Woodcuts are employed to illustrate the definitions, and are placed by the side of the ordinary figures used in demonstrations. It is expected that these woodcuts will be of great value in helping the student to gain clear conceptions of the figures and to cultivate a correct geometrical imagination.

Book IX, Conic Sections, has been carefully revised, and new figures have been substituted for the old ones.

The excellent cuts of the Solid Geometry are largely due to Miss M. Gertrude Cross, of Boston, Mass.

G. A. WENTWORTH.

EXETER, N. H., 1899.

CONTENTS.

SOLID GEOMETRY.

34. GENERAL AXIOMS.

1. Magnitudes which are equal to the same magnitude, or equal magnitudes, are equal to each other.
2. If equals are added to equals, the sums are equal.
3. If equals are taken from equals, the remainders are equal.
4. If equals are added to unequals, the sums are unequal in the same order; if unequals are added to unequals in the same order, the sums are unequal in that order.
5. If equals are taken from unequals, the remainders are unequal in the same order; if unequals are taken from equals, the remainders are unequal in the reverse order.
6. The doubles of the same magnitude, or of equal magnitudes, are equal; and the doubles of unequals are unequal.
7. The halves of the same magnitude, or of equal magnitudes, are equal; and the halves of unequals are unequal.
8. The whole is greater than any of its parts.
9. The whole is equal to the sum of all its parts.

35. SYMBOLS AND ABBREVIATIONS.

$>$ is (or are) greater than.	Def. . . . definition.
$<$ is (or are) less than.	Ax. . . . axiom.
$\backsimeq$ is (or are) equivalent to.	Hyp. . . hypothesis.
$\therefore$ therefore.	Cor. . . . corollary.
$\perp$ perpendicular.	Scho. . . scholium.
$\perp$s perpendiculars.	Ex. . . . exercise.
$\parallel$ parallel. $\parallel$s parallels.	Adj. . . . adjacent.
$\angle$ angle. $\angle$s angles.	Iden. . . identical.
$\triangle$ triangle. $\triangle$s triangles.	Const. . . construction.
$\square$ parallelogram.	Sup. . . . supplementary.
$\square$s parallelograms.	Ext. . . . exterior.
$\odot$ circle. $\odot$s circles.	Int. . . . interior.
rt. right. st. straight.	Alt. . . . alternate.

Q.E.D. stands for quod erat demonstrandum, *which was to be proved.*

Q.E.F. stands for quod erat faciendum, *which was to be done.*

The signs $+$, $-$, $\times$, $\div$, $=$, have the same meaning as in Algebra.

REFERENCES TO PLANE GEOMETRY.

83. At a given point in a given line there can be but one perpendicular to the line.

84. The complements of the same angle or of equal angles are equal.

88. The sum of all the angles about a point in a plane is equal to a perigon, or two straight angles.

93. If one straight line intersects another straight line, the vertical angles are equal.

95. Two straight lines drawn from a point in a perpendicular to a given line, cutting off on the given line equal segments from the foot of the perpendicular, are equal and make equal angles with the perpendicular.

97. The perpendicular is the shortest line that can be drawn to a straight line from an external point.

100. The sum of two lines drawn from a point to the extremities of a straight line is greater than the sum of two other lines similarly drawn, but included by them.

103. Two parallel lines are lines that lie in the same plane and cannot meet however far they are produced.

104. Two straight lines in the same plane perpendicular to the same straight line are parallel.

105. Axiom. Through a given point only one straight line can be drawn parallel to a given straight line.

107. If a straight line is perpendicular to one of two parallel lines, it is perpendicular to the other also.

111. When two straight lines in the same plane are cut by a transversal, if the alternate-interior angles are equal, the two straight lines are parallel.

114. When two straight lines in a plane are cut by a transversal, if the exterior-interior angles are equal, these two straight lines are parallel.

117. A triangle is a portion of a plane bounded by three straight lines. The bounding lines are called the sides of the triangle, and their sum is called the perimeter; the angles included by the sides are called the angles of the triangle, and the vertices of these angles, the vertices of the triangle.

128. The homologous sides and the homologous angles of equal triangles are equal.

138. The sum of two sides of a triangle is greater than the third side, and their difference is less than the third side.

141. Two right triangles are equal if the hypotenuse and an acute angle of the one are equal, respectively, to the hypotenuse and an acute angle of the other.

142. Two right triangles are equal if a leg and an acute angle of the one are equal, respectively, to a leg and the homologous acute angle of the other.

143. Two triangles are equal if two sides and the included angle of the one are equal, respectively, to two sides and the included angle of the other.

144. Two right triangles are equal if their legs are equal, each to each.

145. In an isosceles triangle the angles opposite the equal sides are equal.

147. If two angles of a triangle are equal, the sides opposite the equal angles are equal, and the triangle is isosceles.

149. The perpendicular from the vertex to the base of an isosceles triangle bisects the base, and bisects the vertical angle of the triangle.

150. Two triangles are equal if the three sides of the one are equal, respectively, to the three sides of the other.

151. Two right triangles are equal if a leg and the hypotenuse of the one are equal, respectively, to a leg and the hypotenuse of the other.

155. If two sides of a triangle are equal, respectively, to two sides of another, but the third side of the first triangle is greater than the third side of the second, then the angle opposite the third side of the first triangle is greater than the angle opposite the third side of the second.

160. The perpendicular bisector of a given line is the locus of points equidistant from the extremities of the line.

161. Two points each equidistant from the extremities of a line determine the perpendicular bisector of the line.

166. A parallelogram is a quadrilateral which has its opposite sides parallel.

176. Two angles whose sides are parallel, each to each, are either equal or supplementary.

178. The opposite sides of a parallelogram are equal.

179. A diagonal divides a parallelogram into two equal triangles.

180. Parallel lines comprehended between parallel lines are equal.

183. If two sides of a quadrilateral are equal and parallel, then the other two sides are equal and parallel, and the figure is a parallelogram.

185. Two parallelograms are equal, if two sides and the included angle of the one are equal, respectively, to two sides and the included angle of the other.

186. Two rectangles having equal bases and altitudes are equal.

187. If three or more parallels intercept equal parts on one transversal, they intercept equal parts on every transversal.

188. If a line is parallel to the base of a triangle and bisects one side, it bisects the other side also.

189. The line which joins the middle points of two sides of a triangle is parallel to the third side, and is equal to half the third side.

190. The median of a trapezoid is parallel to the bases, and is equal to half the sum of the bases.

203. Two polygons may be mutually equiangular without being mutually equilateral.

And, except in the case of triangles, two polygons may be mutually equilateral without being mutually equiangular.

If two polygons are mutually equilateral and mutually equiangular, they are equal, for they can be made to coincide.

207. The exterior angles of a polygon, made by producing each of its sides in succession, are together equal to four right angles.

208. Two points are said to be symmetrical with respect to a third point, called the centre of symmetry, if this third point bisects the straight line which joins them.

209. A figure is symmetrical with respect to a point as a centre of symmetry, if the point bisects every straight line drawn through it and terminated by the boundary of the figure.

213. If a figure is symmetrical with respect to two axes perpendicular to each other, it is symmetrical with respect to their intersection as a centre.

216. A circle is a portion of a plane bounded by a curved line, all points of which are equally distant from a point within called the centre. The bounding line is called the circumference of the circle.

217. A radius is a straight line from the centre to the circumference; and a diameter is a straight line through the centre, with its ends in the circumference.

By the definition of a circle, all its radii are equal. All its diameters are equal, since a diameter is equal to two radii.

237. In the same circle or in equal circles, equal arcs subtend equal central angles; and of two unequal arcs the greater subtends the greater central angle.

241. In the same circle or in equal circles, equal arcs are subtended by equal chords; and of two unequal arcs the greater is subtended by the greater chord.

249. In the same circle or in equal circles, equal chords are equally distant from the centre. CONVERSELY: Chords equally distant from the centre are equal.

253. A straight line perpendicular to a radius at its extremity is a tangent to the circle.

254. A tangent to a circle is perpendicular to the radius drawn to the point of contact.

261. The tangents to a circle drawn from an external point are equal, and make equal angles with the line joining the point to the centre.

264. If two circles intersect each other, the line of centres is perpendicular to their common chord at its middle point.

269. Two quantities of the same kind that cannot both be expressed in integers in terms of a common unit, are said to be incommensurable, and the exact value of their ratio cannot

be found. But by taking the unit sufficiently small, an approximate value can be found that shall differ from the true value of the ratio by less than any assigned value, however small.

275. TEST FOR A LIMIT. In order to prove that a variable approaches a constant as a limit, it is necessary to prove that the difference between the variable and the constant:

1. Can be made less than any assigned value, however small.
2. Cannot be made absolutely equal to zero.

278. THEOREM. The limit of the sum of a finite number of variables $x, y, z, \cdots$ is equal to the sum of their respective limits $a, b, c, \cdots$.

279. THEOREM. If the limit of a variable x is not zero, and if k is any finite constant, the limit of the product kx is equal to the limit of x multiplied by k.

280. The quotient of the limit of a variable x by any finite constant k is the limit of x divided by k.

281. THEOREM. The limit of the product of two or more variables is the product of their respective limits, provided no one of these limits is zero.

283. The limit of the nth root of a variable is the nth root of its limit.

284. THEOREM. If two variables are constantly equal, and each approaches a limit, the limits are equal.

285. THEOREM. If two variables have a constant ratio, and each approaches a limit that is not zero, the limits have the same ratio.

288. A circumference is divided into 360 equal parts, called degrees; and therefore a unit angle at the centre intercepts a

unit arc on the circumference. Hence, the numerical measure of a central angle expressed in terms of the unit angle is equal to the numerical measure of its intercepted arc expressed in terms of the unit arc. This must be understood to be the meaning when it is said that

A central angle is measured by its intercepted arc.

330. If four quantities are in proportion, they are in proportion by alternation; that is, the first term is to the third as the second is to the fourth.

333. If four quantities are in proportion, they are in proportion by division; that is, the difference of the first two terms is to the second term as the difference of the last two terms is to the fourth term.

335. In a series of equal ratios, the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.

338. Like powers of the terms of a proportion are in proportion.

342. If a line is drawn through two sides of a triangle parallel to the third side, it divides those sides proportionally.

343. One side of a triangle is to either part cut off by a straight line parallel to the base as the other side is to the corresponding part.

351. Similar polygons are polygons that have their homologous angles equal, and their homologous sides proportional.

354. Two mutually equiangular triangles are similar.

357. If two triangles have an angle of the one equal to an angle of the other, and the including sides proportional, they are similar.

358. If two triangles have their sides respectively proportional, they are similar.

359. Two triangles which have their sides respectively parallel, or respectively perpendicular, are similar.

365. If two polygons are similar, they are composed of the same number of triangles, similar each to each, and similarly placed.

367. If in a right triangle a perpendicular is drawn from the vertex of the right angle to the hypotenuse:

1. The triangles thus formed are similar to the given triangle, and to each other.
2. The perpendicular is the mean proportional between the segments of the hypotenuse.
3. Each leg of the right triangle is the mean proportional between the hypotenuse and its adjacent segment.

370. The perpendicular from any point in the circumference to the diameter of a circle is the mean proportional between the segments of the diameter.

The chord drawn from any point in the circumference to either extremity of the diameter is the mean proportional between the diameter and the adjacent segment.

371. The sum of the squares of the two legs of a right triangle is equal to the square of the hypotenuse.

372. The square of either leg of a right triangle is equal to the difference of the square of the hypotenuse and the square of the other leg.

381. If from a point without a circle a secant and a tangent are drawn, the tangent is the mean proportional between the whole secant and its external segment.

398. The area of a rectangle is equal to the product of its base by its altitude.

400. The area of a parallelogram is equal to the product of its base by its altitude.

401. Parallelograms having equal bases and equal altitudes are equivalent.

403. The area of a triangle is equal to half the product of its base by its altitude.

405. Triangles having equal bases are to each other as their altitudes; triangles having equal altitudes are to each other as their bases; any two triangles are to each other as the products of their bases by their altitudes.

410. The areas of two triangles which have an angle of the one equal to an angle of the other are to each other as the products of the sides including the equal angles.

412. The areas of two similar polygons are to each other as the squares of any two homologous sides.

414. The homologous sides of two similar polygons have the same ratio as the square roots of their areas.

454. The circumference of a circle is the limit which the perimeters of regular inscribed polygons and of similar circumscribed polygons approach, if the number of sides of the polygons is indefinitely increased; and the area of a circle is the limit which the areas of these polygons approach.

463. The area of a circle is equal to π times the square of its radius.

SOLID GEOMETRY.

Book VI.

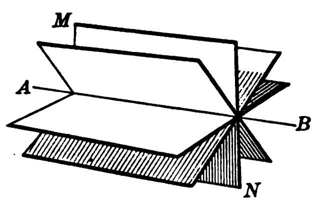
LINES AND PLANES IN SPACE.

DEFINITIONS.

492. Def. A **plane** is a surface such that a straight line joining any two points in it lies wholly in the surface. A plane is understood to be indefinite in extent; but is usually represented by a parallelogram lying in the plane.

493. Def. A plane is said to be **determined** by given lines or points, if no other plane can contain the given lines or points without coinciding with that plane.

494. Cor. 1. *One straight line does not determine a plane.*



For a plane can be made to turn about any straight line AB in it, and thus assume as many different positions as we please.

495. Cor. 2. *A straight line and a point not in the line determine a plane.*

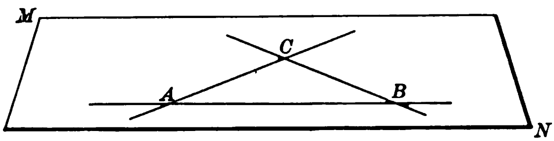
For, if a plane containing a straight line AB and any point C not in AB is made to revolve either way about AB, it will no longer contain the point C.

496. Cor. 3. *Three points not in a straight line determine a plane.*

For by joining two of the points we have a straight line and a point without it, and these determine the plane. § 495

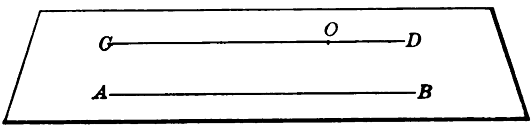
497. Cor. 4. *Two intersecting lines determine a plane.*

For the plane containing one of these lines and any point of the other line not the point of intersection is determined. § 495



498. Cor. 5. *Two parallel lines determine a plane.*

For two parallel lines lie in a plane (§ 103), and a plane containing either parallel and a point in the other is determined. § 495



499. Def. When we suppose a plane to be drawn through given points or lines, we are said to **pass** the plane through the given points or lines.

500. Def. When a straight line is drawn from a point to a plane, its intersection with the plane is called its **foot**.

501. Def. A straight line is **perpendicular** to a plane, if it is perpendicular to every straight line drawn through its foot in the plane; and the plane is perpendicular to the line.

502. Def. A straight line and a plane are **parallel** if they cannot meet, however far both are produced.

503. Def. A straight line neither perpendicular nor parallel to a plane is said to be **oblique** to the plane.

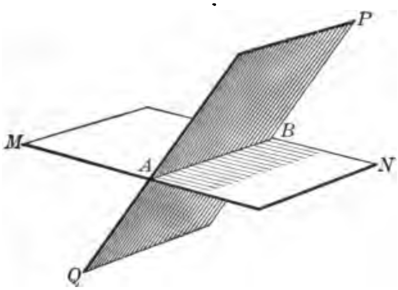
504. Def. Two planes are **parallel** if they cannot meet, however far they are produced.

505. Def. The **intersection of two planes** contains all the points common to the two planes.

LINES AND PLANES.

Proposition I. Theorem.

506. *If two planes cut each other, their intersection is a straight line.*



Let MN and PQ be two planes which cut one another.

To prove that their intersection is a straight line.

Proof. Let A and B be two points common to the two planes.

Draw a straight line through the points A and B.

Then the straight line AB lies in both planes. § 492

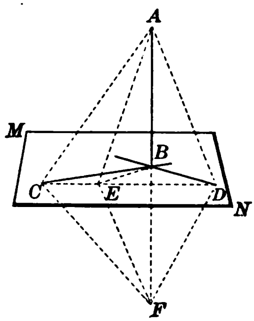
No point not in the line AB can be in both planes; for one plane, and only one, can contain a straight line and a point without the line. § 495

Therefore, the straight line through A and B contains all the points common to the two planes, and is consequently the intersection of the planes. § 505

Q. E. D.

PROPOSITION II. THEOREM.

507. *If a straight line is perpendicular to each of two other straight lines at their point of intersection, it is perpendicular to the plane of the two lines.*



Let AB be perpendicular to BC and BD at B.

To prove that AB is ⊥ to the plane MN of these lines.

Proof. Through B draw in MN *any* other straight line BE, and draw CD cutting BC, BE, BD, at C, E, and D.

Prolong AB to F, making BF equal to AB, and join A and F to each of the points C, E, and D.

Then BC and BD are each ⊥ to AF at its middle point.

$\therefore AC = FC$, and $AD = FD$. § 160

$\therefore \triangle ACD = \triangle FCD$. § 150

$\therefore \angle ACD = \angle FCD$. § 128

That is, $\angle ACE = \angle FCE$.

Hence, the $\triangle$ ACE and FCE are equal. § 143

For $AC = FC$, $CE = CE$, and $\angle ACE = \angle FCE$.

$\therefore AE = FE$; and BE is ⊥ to AF at B. § 161

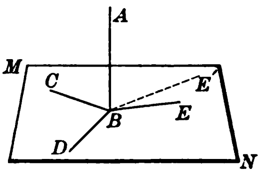
$\therefore AB$ is ⊥ to any and hence every line in MN through B.

$\therefore AB$ is ⊥ to MN. § 501

Q. E. D.

PROPOSITION III. THEOREM.

508. *All the perpendiculars that can be drawn to a straight line at a given point lie in a plane which is perpendicular to the line at the given point.*



Let the plane MN be perpendicular to AB at B.

To prove that BE, any $\perp$ to AB at B, lies in MN.

Proof. Let the plane containing AB and BE intersect MN in the line BE'; then AB is $\perp$ to BE'. § 501

Since in the plane ABE only one $\perp$ can be drawn to AB at B (§ 83), BE and BE' coincide, and BE lies in MN.

Hence, every $\perp$ to AB at B lies in the plane MN. Q. E. D.

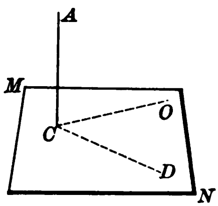
509. COR. 1. *At a given point in a straight line one plane perpendicular to the line can be drawn, and only one.*

510. COR. 2. *Through a given external point, one plane can be drawn perpendicular to a line, and only one.*

Let AC be the line, and O the point.

Draw $OC \perp$ to AC, and $CD \perp$ to AC. Then CO and CD determine a plane through $O \perp$ to AC.

Only one such plane can be drawn; for only one $\perp$ can be drawn to AC from the point O. § 96



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Proposition IV. Theorem.

511. *Through a given point there can be one perpendicular to a given plane, and only one.*

Insert BD.

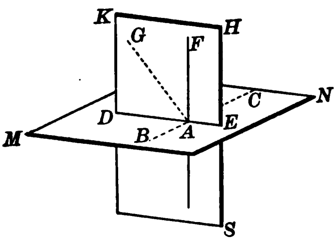


Fig. 1.

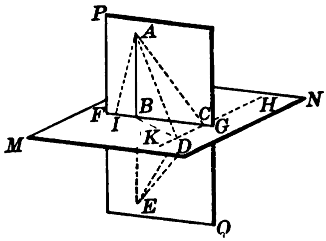


Fig. 2.

Case 1. *When the given point is in the given plane.*

Let A be the given point in the plane MN (Fig. 1).

To prove that there can be one perpendicular to the plane MN at A, and only one.

Proof. Through *A* draw in *MN* any line *BC*, and pass through *A* a plane *KS* ⊥ to *BC*, cutting *MN* in *DE*.

At *A* erect in the plane *KS* the line *AF* ⊥ to *DE*.

The line *BC*, being ⊥ to the plane *KS* by construction, is ⊥ to *AF*, which passes through its foot in the plane. § 501

That is, *AF* is ⊥ to *BC*; and as it is ⊥ to *DE* by construction, it is ⊥ to the plane *MN*. § 507

Moreover, any other line *AG* drawn from *A* is oblique to *MN*. For *AF* and *AG* intersecting in *A* determine a plane *KS*, which cuts *MN* in the straight line *DE*; and as *AF* is ⊥ to *MN*, it is ⊥ to *DE* (§ 501); hence, *AG* is oblique to *DE* (§ 83), and therefore to *MN*. § 503

Therefore, *AF* is the only ⊥ to *MN* at the point *A*.

CASE 2. *When the given point is without the given plane.*

Let A be the given point, and MN the given plane (Fig. 2).

To prove that there can be one perpendicular from A to MN, and only one.

Proof. Draw in MN any line HK, and pass through A a plane $PQ \perp$ to HK, cutting MN in FG, and HK in C.

Let fall from A, in the plane PQ, a $\perp$ AB upon FG.

Draw in the plane MN any other line BD from B, intersecting HK in D.

Prolong AB to E, making BE equal to AB,

and join A and E to each of the points C and D.

Since DC is $\perp$ to PQ by construction, and CA and CE lie in PQ, the $\angle$s DCA and DCE are right angles. § 501

The rt. $\triangle$s DCA and DCE are equal. § 144

For DC is common; and $CA = CE$. § 160

$\therefore DA = DE$. § 128

$\therefore BD$ is $\perp$ to AE at B. § 161

That is, AB is $\perp$ to BD, any straight line drawn in MN through its foot, and therefore is $\perp$ to MN. § 501

Moreover, every other straight line AI drawn from A to the plane is oblique to MN. For the lines AB and AI determine a plane PQ which cuts the plane MN in the line FG.

The line AB, being $\perp$ to the plane MN, is $\perp$ to FG. § 501

$\therefore AI$ is oblique to FG, and consequently to MN. § 503

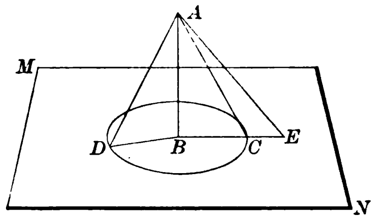
Therefore, AB is the only $\perp$ from A to MN. Q.E.D.

512. COR. *The perpendicular is the shortest line from a point to a plane.*

513. DEF. The **distance from a point to a plane** is the length of the perpendicular from the point to the plane.

PROPOSITION V. THEOREM.

514. *Oblique lines drawn from a point to a plane, meeting the plane at equal distances from the foot of the perpendicular, are equal; and of two oblique lines meeting the plane at unequal distances from the foot of the perpendicular the more remote is the greater.*



Let AC and AD cut off the equal distances BC and BD from the foot of the perpendicular AB, and let AD and AE cut off the unequal distances BD and BE, and BE be greater than BD.

To prove that $AC = AD$, *and* $AE > AD$.

Proof. The rt. $\triangle s$ ABC and ABD are equal. § 144

For AB is common, and $BC = BD$. Hyp.

$\therefore AC = AD$. § 128

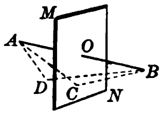
The rt. $\triangle s$ ABE, ABC have AB common, and $BE > BC$.

$\therefore AE > AC$ (§ 101), and hence $AE > AD$. Q.E.D.

515. COR. 1. *Equal oblique lines from a point to a plane meet the plane at equal distances from the foot of the perpendicular; and of two unequal lines the greater meets the plane at the greater distance from the foot of the perpendicular.*

516. COR. 2. *The locus of a point in space equidistant from all points in the circumference of a circle is a straight line through the centre, perpendicular to the plane of the circle.*

517. Cor 3. *The locus of a point in space equidistant from the extremities of a straight line is the plane perpendicular to this line at its middle point.*

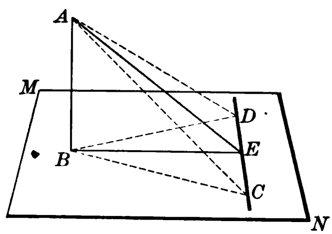


For any point C in this plane lies in a $\perp$ to AB at O, its middle point; hence, CA and CB are equal. § 160

And any point D without the plane MN cannot lie in a $\perp$ to AB at O, and hence is unequally distant from A and B. § 160

Proposition VI. Theorem.

518. *If from the foot of a perpendicular to a plane a straight line is drawn at right angles to any line in the plane, the line drawn from its intersection with the line in the plane to any point of the perpendicular is perpendicular to the line of the plane.*



Let AB be a perpendicular to the plane MN, BE a perpendicular from B to any line CD in MN, and EA any line from E to AB.

To prove that AE *is* $\perp$ *to* CD.

Proof. Take EC and ED equal; draw BC, BD, AC, AD.

Now $BC = BD$. § 95

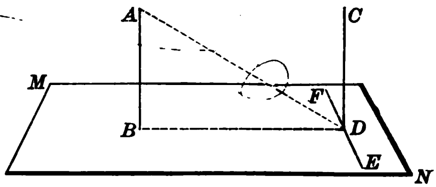
$\therefore AC = AD$. § 514

$\therefore AE$ is $\perp$ to CD. § 161

Q. E. D.

Proposition VII. Theorem.

519. *Two straight lines perpendicular to the same plane are parallel.*



Let AB be perpendicular to MN at B, and CD to MN at D.

To prove that AB and CD are parallel.

Proof. From A any external point in AB draw AD and BD.

Through D draw EF in the plane MN $\perp$ to BD.

Then CD is $\perp$ to EF (§ 501), and AD is $\perp$ to EF. § 518

Therefore, CD, AD, and BD lie in the same plane. § 508

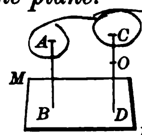
Also the line AB lies in this plane. § 492

But AB and CD are both $\perp$ to BD. § 501

Therefore, AB and CD are parallel. § 104

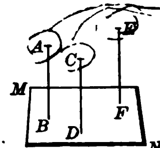
Q. E. D.

520. Cor. 1. *If one of two parallel lines is perpendicular to a plane, the other is also perpendicular to the plane.*



For if through any point O of CD a line is drawn $\perp$ to MN, it is $\parallel$ to AB (§ 519), and CD coincides with this $\perp$ and is $\perp$ to MN. § 105

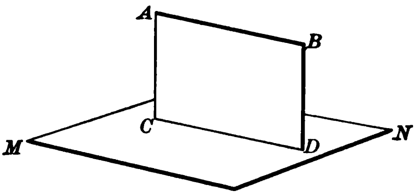
521. Cor. 2. *If two straight lines are parallel to a third straight line, they are parallel to each other.*



For a plane MN $\perp$ to CD is $\perp$ to AB and EF. § 520

PROPOSITION VIII. THEOREM.

522. *If two straight lines are parallel, every plane containing one of the lines, and only one, is parallel to the other line.*



Let AB and CD be two parallel lines, and MN any plane containing CD but not AB.

To prove that AB and MN are parallel.

Proof. The lines AB and CD are in the same plane, § 103
and this plane intersects the plane MN in the line CD. Hyp.

Therefore, if AB meets the plane MN at all, the point of meeting must be in the line CD.

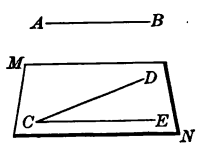
But since AB is $\parallel$ to CD, AB cannot meet CD.

Therefore, AB cannot meet the plane MN.

Hence, AB is $\parallel$ to MN. § 502

Q. E. D.

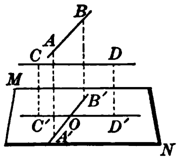
523. COR. 1. *Through either of two straight lines not in the same plane one plane, and only one, can be passed parallel to the other.*



For if AB and CD are the lines, and we pass a plane through CD and the line CE drawn $\parallel$ to AB, the plane MN determined by CD and CE is $\parallel$ to AB. § 522

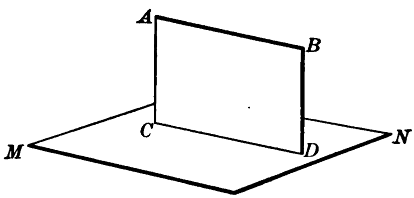
524. Cor. 2. *Through a given point a plane can be passed parallel to any two given straight lines in space.*

For if O is the given point, and AB and CD the given lines, by drawing through O a line $A'B'$ ‖ to AB, and also a line $C'D'$ ‖ to CD, we shall have two lines $A'B'$ and $C'D'$ which determine a plane passing through O and ‖ to each of the lines AB and CD. § 522



Proposition IX. Theorem.

525. *If a straight line is parallel to a plane, the intersection of the plane with any plane passed through the given line is parallel to that line.*



Let the line AB be parallel to the plane MN, and let CD be the intersection of MN with any plane AD passed through AB.

To prove that AB and CD are parallel.

Proof. The lines AB and CD are in the same plane AD.

Since CD lies in the plane MN, if AB meets CD it must meet the plane MN.

But AB is by hypothesis ‖ to MN, and therefore cannot meet it; that is, it cannot meet CD, however far they may be produced.

Hence, AB and CD are parallel. § 103

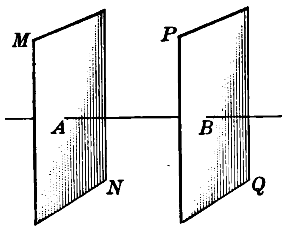
Q. E. D.

526. Cor. *If a given straight line and a plane are parallel, a parallel to the given line drawn through any point of the plane lies in the plane.*

For the plane determined by the given line AB and any point C of the plane cuts MN in a line $CD \parallel$ to AB (§ 525); but through C only one parallel to AB can be drawn (§ 105); therefore, a line drawn through $C \parallel$ to AB coincides with CD, and hence lies in the plane MN.

Proposition X. Theorem.

527. *Two planes perpendicular to the same straight line are parallel.*



Let MN and PQ be two planes perpendicular to the straight line AB.

To prove that MN and PQ are parallel.

Proof. MN and PQ cannot meet.

For if they could meet, we should have two planes from a point of their intersection $\perp$ to the same straight line.

But this is impossible. § 510

Therefore, MN and PQ are parallel. § 504

Q.E.D.

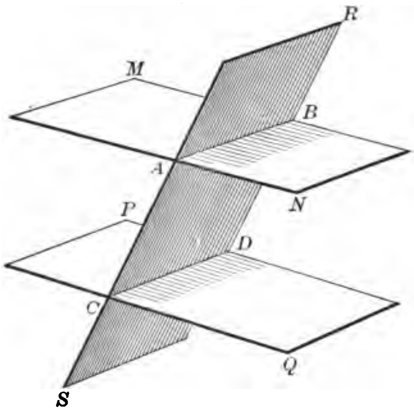
Ex. 604. Find the locus of points in space equidistant from two given parallel planes.

Ex. 605. Find the locus of points in space equidistant from two given points and also equidistant from two given parallel planes.

PROPOSITION XI. THEOREM.

528. *The intersections of two parallel planes by a third plane are parallel lines.*

Trim Shad



Let the parallel planes MN and PQ be cut by RS.

To prove that the intersections AB and CD are parallel.

Proof. AB and CD are in the same plane RS.

They are also in the parallel planes MN and PQ, which cannot meet, however far they extend. § 504

Therefore, AB and CD cannot meet, and are parallel. § 103

Q. E. D.

529. COR. 1. *Parallel lines included between parallel planes are equal.*

For if the lines AC and BD are parallel, the plane of these lines will intersect MN and PQ in the parallel lines AB and CD. § 528

$\therefore$ $ABDC$ is a parallelogram. § 166

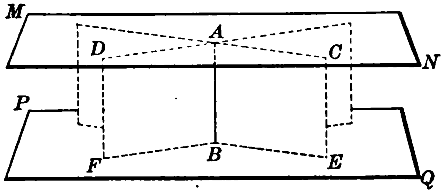
$\therefore$ AC and BD are equal. § 178

530. COR. 2. *Two parallel planes are everywhere equally distant.*

For ⊥s dropped from *any* points in MN to PQ measure the distances of these points from PQ. But these ⊥s are parallel (§ 519), and hence equal (§ 529). Therefore, *all* points in MN are equidistant from PQ.

PROPOSITION XII. THEOREM.

531. *A straight line perpendicular to one of two parallel planes is perpendicular to the other also.*



Let AB be perpendicular to MN and PQ parallel to MN.

To prove that AB is perpendicular to PQ.

Proof. Pass through the line AB any two planes intersecting MN in the lines AC and AD, and PQ in BE and BF. Then AC and AD are ∥ to BE and BF, respectively. § 528

But AB is ⊥ to AC and AD. § 501

∴ AB is ⊥ to their parallels BE and BF. § 107

Therefore, AB is ⊥ to PQ. § 507

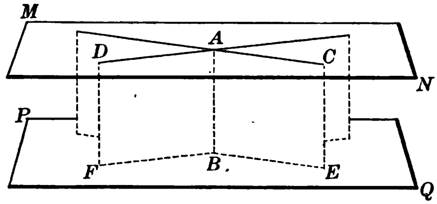
Q. E. D.

532. COR. *Through a given point one plane, and only one, can be drawn parallel to a given plane.*

For if a line is drawn from A ⊥ to PQ, a plane passing through A ⊥ to this line is ∥ to PQ (§ 527); and since through a point in a line only one plane can be drawn ⊥ to the line (§ 509), only one plane can be drawn through A ∥ to PQ.

PROPOSITION XIII. THEOREM.

533. *If two intersecting straight lines are each parallel to a plane, the plane of these lines is parallel to that plane.*



Let AC and AD be each parallel to the plane PQ, and let MN be the plane passed through AC and AD.

To prove that MN is parallel to PQ.

Proof. Draw $AB \perp$ to PQ.

Pass a plane through AB and AC intersecting PQ in BE, and a plane through AB and AD intersecting PQ in BF.

Then AB is $\perp$ to BE and BF. § 501

Also, BE is $\parallel$ to AC, and BF is $\parallel$ to AD. § 525

Therefore, AB is $\perp$ to AC and to AD. § 107

Therefore, AB is $\perp$ to the plane MN. § 507

Hence, MN and PQ are parallel. § 527

Q. E. D.

Ex. 606. Find the locus of all lines drawn through a given point, parallel to a given plane.

Ex. 607. Find the locus of points in a given plane which are equidistant from two given points not in the plane.

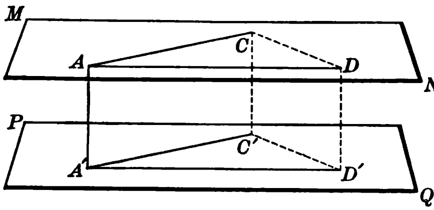
Ex. 608. Find the locus of a point in space equidistant from three given points not in a straight line.

Ex. 609. Find a point in a plane such that the sum of its distances from two given points on the same side of the plane shall be a minimum.

?

Proposition XIV. Theorem.

534. *If two angles not in the same plane have their sides respectively parallel and lying on the same side of the straight line joining their vertices, they are equal, and their planes are parallel.*



Let the angles A and A′ be respectively in the planes MN and PQ, and have AD parallel to A′D′ and AC parallel to A′C′, and lying on the same side of AA′.

?

To prove that $\angle A = \angle A'$, *and that* MN *is* $\parallel$ *to* PQ.

Proof. Take AD and $A'D'$ equal, also AC and $A'C'$ equal.

Draw DD', CC', CD, $C'D'$.

Since AD is equal and $\parallel$ to $A'D'$, the figure $ADD'A'$ is a parallelogram, and AA' is equal and $\parallel$ to DD'. § 183

In like manner AA' is equal and $\parallel$ to CC'.

Also, since CC' and DD' are each $\parallel$ to AA', and equal to AA', they are $\parallel$ and equal.

$\therefore CD = C'D'$. § 183

$\therefore \triangle ADC = \triangle A'D'C'$. § 150

$\therefore \angle A = \angle A'$. § 128

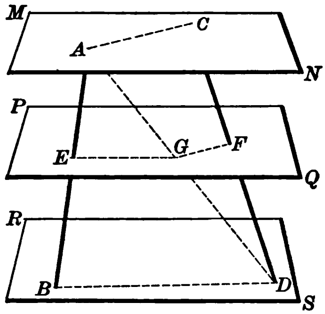
Now PQ is $\parallel$ to each of the lines AC and AD. § 522

Therefore, PQ is $\parallel$ to MN, the plane of these lines. § 533

Q. E. D.

Proposition XV. Theorem.

535. *If two straight lines are intersected by three parallel planes, their corresponding segments are proportional.*



Let AB and CD be intersected by the parallel planes MN, PQ, RS, in the points A, E, B, and C, F, D.

To prove that $AE : EB = CF : FD$.

Proof. Draw AD cutting the plane PQ in G.

Draw AC, BD, EG, and FG.

Then EG is $\parallel$ to BD, and GF is $\parallel$ to AC. § 528

$\therefore AE : EB = AG : GD$, § 342

and $CF : FD = AG : GD$.

$\therefore AE : EB = CF : FD$. Ax. 1

Q. E. D.

Ex. 610. The line AB meets three parallel planes in the points A, E, B; and the line CD meets the same planes in the points C, F, D. If $AE = 6$ inches, $BE = 8$ inches, $CD = 12$ inches, compute CF and FD.

Ex. 611. To draw a perpendicular to a given plane from a given point without the plane.

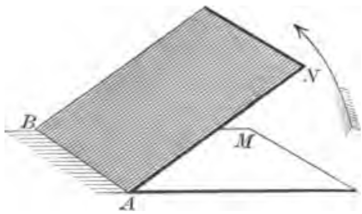
Ex. 612. To erect a perpendicular to a given plane at a given point in the plane.

DIHEDRAL ANGLES.

536. Def. The *opening* between two intersecting planes is called a **dihedral angle.**

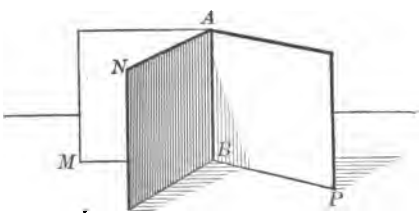
537. Def. The line of intersection *AB* of the planes is the **edge,** the planes *MA* and *NB* are the **faces,** of the dihedral angle.

538. A dihedral angle is designated by its edge, or by its two faces and its edge. Thus, the dihedral angle in the margin may be designated by *AB,* or by *M-AB-N.*



539. In order to have a clear notion of the *magnitude* of the dihedral angle *M-AB-N,* suppose a plane at first in coincidence with the plane *MA* to turn about the edge *AB,* as indicated by the arrow, until it coincides with the plane *NB.* The magnitude of the dihedral angle *M-AB-N* is proportional to the *amount of rotation* of this plane.

540. Def. Two dihedral angles *M-AB-N* and *P-AB-N* are **adjacent** if they have a common edge *AB,* and a common face *NB,* between them.



541. Def. When a plane meets another plane and makes the *adjacent dihedral angles equal,* each of these angles is called a **right dihedral angle.**

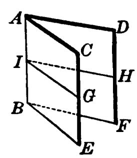
542. Def. A plane is **perpendicular** to another plane if it forms with this second plane a right dihedral angle.

543. Def. Two **vertical dihedral angles** are dihedral angles that have the same edge and the faces of the one are the prolongations of the faces of the other.

544. Def. Dihedral angles are **acute, obtuse, complementary, supplementary**, under the same conditions as plane angles.

545. Def. The **plane angle of a dihedral angle** is the plane angle formed by two straight lines, one in each plane, perpendicular to the edge at the same point.

546. Cor. *The plane angle of a dihedral angle has the same magnitude from whatever point in the edge the perpendiculars are drawn.*



For any two such angles, as CAD, GIH, have their sides respectively parallel (§ 104), and hence are equal. § 534

547. The demonstrations of many properties of dihedral angles are identically the same as the demonstrations of analogous theorems of plane angles.

The following are examples:

1. If a plane meets another plane, it forms with it two adjacent dihedral angles whose sum is equal to two right dihedral angles.

2. If the sum of two adjacent dihedral angles is equal to two right dihedral angles, their exterior faces are in the same plane.

3. If two planes intersect each other, their vertical dihedral angles are equal.

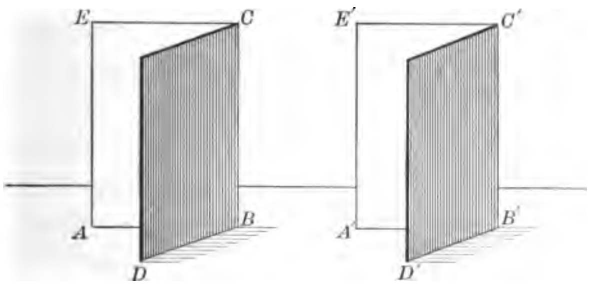
4. If a plane intersects two parallel planes, the alternate-interior dihedral angles are equal; the exterior-interior dihedral angles are equal; the two interior dihedral angles on the same side of the transverse plane are supplementary.

5. When two planes are cut by a third plane, if the alternate-interior dihedral angles are equal, or the exterior-interior dihedral angles are equal, and the edges of the dihedral angles thus formed are parallel, the two planes are parallel.

6. Two dihedral angles whose faces are parallel each to each are either equal or supplementary.

Proposition XVI. Theorem.

548. *Two dihedral angles are equal if their plane angles are equal.*



Let the two plane angles ABD and A′B′D′ of the two dihedral angles D-CB-E and D′-C′B′-E′ be equal.

To prove the dihedral angles D-CB-E and D′-C′B′-E′ equal.

Proof. Apply D'-$C'B'$-E' to D-CB-E, making the plane angle $A'B'D'$ coincide with its equal ABD.

The line $B'C'$ being $\perp$ to the plane $A'B'D'$ will likewise be $\perp$ to the plane ABD at B, and fall on BC, since at B only one $\perp$ can be erected to this plane. § 511

The two planes $A'B'C'$ and ABC, having in common two intersecting lines AB and BC, coincide. § 497

In like manner the planes $D'B'C'$ and DBC coincide.

Therefore, the two dihedral angles coincide and are equal.

Q. E. D.

PROPOSITION XVII. THEOREM.

549. *Two dihedral angles have the same ratio as their plane angles.*

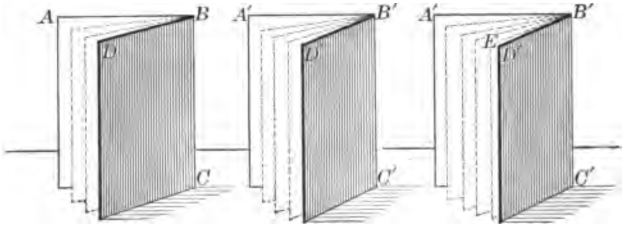


FIG. 1. FIG. 2. FIG. 3.

Let A-BC-D and A′-B′C′-D′ be two dihedral angles, and let their plane angles be ABD and A′B′D′, respectively.

To prove that $A'\text{-}B'C'\text{-}D' : A\text{-}BC\text{-}D = \angle A'B'D' : \angle ABD$.

CASE 1. *When the plane angles are commensurable.*

Proof. Suppose the $\angle$s ABD and $A'B'D'$ (Figs. 1 and 2) have a common measure, which is contained m times in $\angle ABD$ and n times in $\angle A'B'D'$.

Then $\angle A'B'D' : \angle ABD = n : m$.

Apply this measure to $\angle ABD$ and $\angle A'B'D'$, and through the lines of division and the edges BC and $B'C'$ pass planes.

These planes divide $A\text{-}BC\text{-}D$ into m parts, and $A'\text{-}B'C'\text{-}D'$ into n parts, equal each to each. § 548

Therefore, $A'\text{-}B'C'\text{-}D' : A\text{-}BC\text{-}D = n : m$.

Therefore, $A'\text{-}B'C'\text{-}D' : A\text{-}BC\text{-}D = \angle A'B'D' : \angle ABD$. Ax. 1

CASE 2. *When the plane angles are incommensurable.*

Proof. Divide the $\angle ABD$ into any number of equal parts, and apply one of these parts to the $\angle A'B'D'$ (Figs. 1 and 3) as a unit of measure.

Since $\angle ABD$ and $\angle A'B'D'$ are incommensurable, a certain number of these parts will form the $\angle A'B'E$, leaving a remainder $\angle EB'D'$, less than one of the parts.

Pass a plane through $B'E$ and $B'C'$.

Since the plane angles of the dihedral angles $A\text{-}BC\text{-}D$ and $A'\text{-}B'C'\text{-}E$ are commensurable,

$$A'\text{-}B'C'\text{-}E : A\text{-}BC\text{-}D = \angle A'B'E : \angle ABD. \qquad \text{Case 1}$$

By increasing the number of equal parts into which $\angle ABD$ is divided, we can diminish at pleasure the magnitude of each part, and therefore make $\angle EB'D'$ less than any assigned value, however small, since $\angle EB'D'$ is always less than one of the equal parts into which $\angle ABD$ is divided.

But we cannot make $\angle EB'D'$ equal to zero, since by hypothesis $\angle ABD$ and $\angle A'B'D'$ are incommensurable. § 269

Therefore, $\angle EB'D'$ approaches zero as a limit, if the number of parts into which $\angle ABD$ is divided is indefinitely increased; and the corresponding dihedral angle $E\text{-}B'C'\text{-}D'$ approaches zero as a limit. § 275

Therefore, $\angle A'B'E$ approaches $\angle A'B'D'$ as a limit, § 271

and $A'\text{-}B'C'\text{-}E$ approaches $A'\text{-}B'C'\text{-}D'$ as a limit.

Hence, $\dfrac{\angle A'B'E}{\angle ABD}$ approaches $\dfrac{\angle A'B'D'}{\angle ABD}$ as a limit, § 280

and $\dfrac{A'\text{-}B'C'\text{-}E}{A\text{-}BC\text{-}D}$ approaches $\dfrac{A'\text{-}B'C'\text{-}D'}{A\text{-}BC\text{-}D}$ as a limit. § 280

But $\dfrac{\angle A'B'E}{\angle ABD}$ is constantly equal to $\dfrac{A'\text{-}B'C'\text{-}E}{A\text{-}BC\text{-}D}$, Case 1

as $\angle EB'D'$ varies in value and approaches zero as a limit.

Therefore, the limits of these variables are equal. § 284

That is,

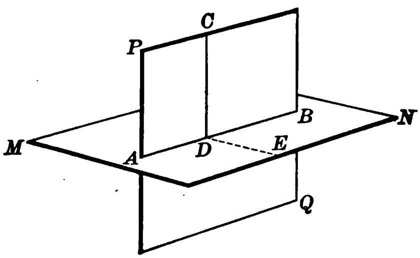
$$\frac{A'\text{-}B'C'\text{-}D'}{A\text{-}BC\text{-}D} = \frac{\angle A'B'D'}{\angle ABD}.$$

Q.E.D.

550. Cor. *The plane angle of a dihedral angle may be taken as the measure of the dihedral angle.*

PROPOSITION XVIII. THEOREM.

551. *If two planes are perpendicular to each other, a straight line drawn in one of them perpendicular to their intersection is perpendicular to the other plane.*



Let the plane PQ be perpendicular to MN, and let CD be drawn in PQ perpendicular to AB, the intersection of PQ and MN.

To prove that CD is perpendicular to MN.

Proof. In the plane *MN* draw *DE* ⊥ to *AB* at *D*.

Then *CDE* is the measure of the right dihedral angle *P-AB-N*, and is therefore a right angle. § 550

But, by hypothesis, *CDA* is a right angle.

Therefore, *CD* is ⊥ to *DA* and *DE* at their point of intersection, and consequently ⊥ to their plane *MN*. § 507

Q. E. D.

552. COR. 1. *If two planes are perpendicular to each other, a perpendicular to one of them at any point of their intersection will lie in the other plane.*

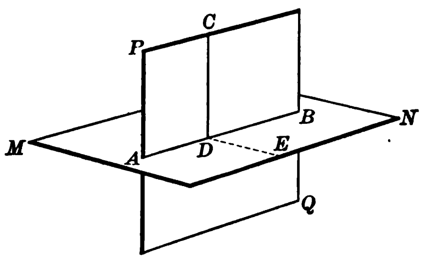
For a line *CD* drawn in the plane *PAB* ⊥ to *AB* at the point *D* will be ⊥ to *MN* (§ 551). But at the point *D* only one ⊥ can be drawn to *MN* (§ 511). Therefore, a ⊥ to *MN* erected at *D* will coincide with *CD* and lie in the plane *PAB*.

553. Cor. 2. *If two planes are perpendicular to each other, a perpendicular to one of them from any point of the other will lie in the other plane.*

For a line CD drawn in the plane PAB from the point $C \perp$ to AB will be $\perp$ to MN (§ 551). But from the point C only one $\perp$ can be drawn to MN (§ 511). Therefore, a $\perp$ to MN drawn from C will coincide with CD and lie in PAB.

Proposition XIX. Theorem.

554. *If a straight line is perpendicular to a plane, every plane passed through this line is perpendicular to the plane.*



Let CD be perpendicular to MN, and PQ be any plane passed through CD intersecting MN in AB.

To prove that PQ is perpendicular to the plane MN.

Proof. Draw DE in the plane $MN \perp$ to AB.

Since CD is $\perp$ to MN, it is $\perp$ to AB. § 501

Therefore, $\angle CDE$ is the measure of $P\text{-}AB\text{-}N$. § 550

But $\angle CDE$ is a right angle. § 501

Therefore, PQ is $\perp$ to MN. § 542

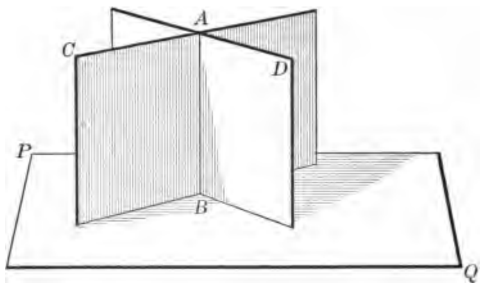
Q. E. D.

555. Cor. *A plane perpendicular to the edge of a dihedral angle is perpendicular to each of its faces.*

PROPOSITION XX. THEOREM.

556. *If two intersecting planes are each perpendicular to a third plane, their intersection is also perpendicular to that plane.*

Trim-shad.



Let the planes BD and BC intersecting in the line AB be perpendicular to the plane PQ.

To prove that AB is perpendicular to the plane PQ.

Proof. A $\perp$ erected to PQ at B, a point common to the three planes, will lie in the two planes BC and BD. § 552

And since this $\perp$ lies in both the planes BC and BD, it must coincide with their intersection AB. § 506

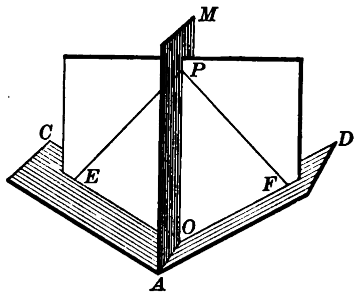
$\therefore$ AB is $\perp$ to the plane PQ. Q.E.D.

557. COR. 1. *If a plane is perpendicular to each of two intersecting planes, it is perpendicular to their intersection.*

558. COR. 2. *If a plane is perpendicular to each of two planes that include a right dihedral angle, the intersection of any two of these planes is perpendicular to the third plane, and each of the three intersections is perpendicular to the other two.*

Proposition XXI. Theorem.

559. *Every point in a plane which bisects a dihedral angle is equidistant from the faces of the angle.*



Let the plane AM bisect the dihedral angle formed by the planes AD and AC; and let PE and PF be perpendiculars drawn from any point P in the plane AM to the planes AC and AD.

To prove that $PE = PF$.

Proof. Through PE and PF pass a plane intersecting the planes AC, AD, and AM in the lines OE, OF, and PO.

The plane PEF is $\perp$ to AC and to AD. § 554

Hence, the plane PEF is $\perp$ to their intersection AO. § 557

$\therefore AO$ is $\perp$ to OE, OP, and OF. § 501

$\therefore \angle POE$ is the measure of $M\text{-}AO\text{-}C$, § 550

and $\angle POF$ is the measure of $M\text{-}AO\text{-}D$.

But $M\text{-}AO\text{-}C = M\text{-}AO\text{-}D$. Hyp.

$\therefore \angle POE = \angle POF$.

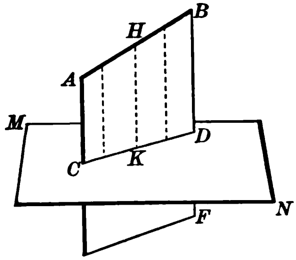
$\therefore$ rt. $\triangle POE =$ rt. $\triangle POF$. § 141

$\therefore PE = PF$. § 128

Q. E. D.

PROPOSITION XXII. THEOREM.

560. *Through a given straight line not perpendicular to a plane, one plane, and only one, can be passed perpendicular to the given plane.*



Let AB be the given line not perpendicular to the plane MN.

To prove that one plane can be passed through AB perpendicular to MN, and only one.

Proof. From any point H of AB draw $HK \perp$ to MN,

and through AB and HK pass a plane AF.

The plane AF is $\perp$ to MN, since it passes through HK, a line $\perp$ to MN. § 554

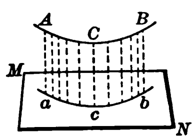
Moreover, if two planes could be passed through $AB \perp$ to the plane MN, their intersection AB would be $\perp$ to MN. § 556

But this is impossible, since AB is by hypothesis not perpendicular to the plane MN.

Hence, through AB only one plane can be passed $\perp$ to MN.

Q. E. D.

561. DEF. The **projection of a point on a plane** is the foot of the perpendicular from the point to the plane.

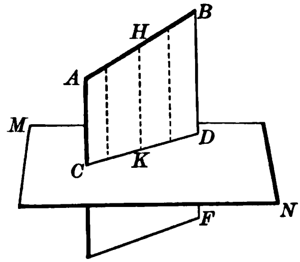


562. DEF. The **projection of a line on a plane** is the locus of the projections of its points on the plane.

?

PROPOSITION XXIII. THEOREM.

563. *The projection of a straight line not perpendicular to a plane upon that plane is a straight line.*



Let AB be the given line, MN the given plane, and CD the projection of AB upon MN. M N

To prove that CD is a straight line.

Proof. From any point H of AB draw $HK \perp$ to ~~CD,~~ and pass a plane AF through HK and AB.

The plane AF is $\perp$ to MN, § 554

and contains all the $\perp$s drawn from AB to MN. § 553

Hence, CD must be the intersection of these two planes.

Therefore, CD is a straight line. § 506

Q. E. D.

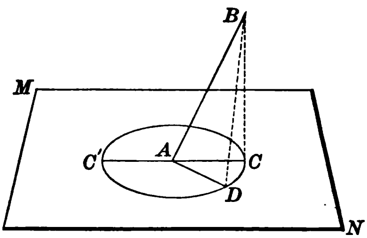
564. COR. *The projection of a straight line perpendicular to a plane upon that plane is a point.*

565. DEF. The plane $ABCD$ is called the **projecting plane** of the line AB upon the plane MN.

566. DEF. The **angle which a line makes with a plane** is the angle which it makes with its projection on the plane; and is called the **inclination** of the line to the plane.

PROPOSITION XXIV. THEOREM.

567. *The acute angle which a straight line makes with its projection upon a plane is the least angle which it makes with any line of the plane.*



Let BA meet the plane MN at A, and let AC be its projection upon the plane MN, and AD any other line drawn through A in the plane.

To prove that $\angle BAC$ *is less than* $\angle BAD$.

Proof. Take AD equal to AC, and draw BD.

In the $\triangle$ BAC and BAD,

	$BA = BA$,	Iden.
	$AC = AD$,	Const.
but	$BC < BD$.	§ 512
	$\therefore \angle BAC$ is less than $\angle BAD$.	§ 155

Q. E. D.

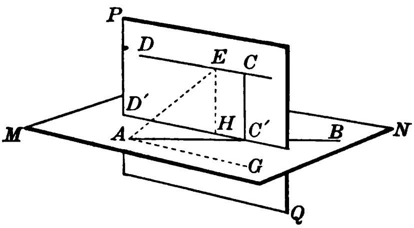
Ex. 613. From a point A, 4 inches from a plane MN, an oblique line AC 5 inches long is drawn to the plane and made to turn around the perpendicular AB dropped from A to the plane. Find the area of the circle described by the point C.

Ex. 614. From a point A, 8 inches from a plane MN, a perpendicular AB is drawn to the plane; with B as centre, and a radius equal to 6 inches, a circle is described in the plane; at any point C of this circumference a tangent CD 24 inches long is drawn. Find the distance from A to D.

Ex. 615. Describe the relative position to a given plane of a line if its projection on the plane is equal to its own length.

Proposition XXV. Theorem.

568. *Between two straight lines not in the same plane, there can be one common perpendicular, and only one.*



Let AB and DC be two lines not in the same plane.

To prove that there can be one common perpendicular between AB and DC, and only one.

Proof. Through any point A of AB draw $AG \parallel$ to DC, and let MN be the plane determined by AB and AG.

Since AG is $\parallel$ to DC, MN is $\parallel$ to DC. § 522

Through DC pass the plane $PQ \perp$ to MN, intersecting the plane MN in $D'C'$. Then $D'C'$ is $\parallel$ to DC. § 525

$D'C'$ must cut AB at some point C', otherwise AB would be $\parallel$ to $D'C'$ (§ 103), and hence $\parallel$ to DC (§ 521). But this is impossible; for AB and DC are not in the same plane. Hyp.

Draw $C'C \perp$ to MN. Then $C'C$ is $\perp$ to AB. § 501

But $C'C$ is in the plane PQ (§ 552) and is $\perp$ to $D'C'$. § 501

$\therefore C'C$ is $\perp$ to DC. § 107

$\therefore C'C$ is $\perp$ to AB and DC.

Again, $C'C$ is the only $\perp$ to both AB and DC. For, if possible, let EA be any other line $\perp$ to AB and DC. Then EA is $\perp$ to AG (§ 107), and hence $\perp$ to MN. § 507

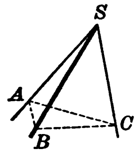
Draw $EH \perp$ to $D'C'$. Then EH is $\perp$ to MN (§ 551), and we have two $\perp$s from E to MN. But this is impossible. § 511

Hence, $C'C$ is the only common $\perp$ to DC and AB. Q. E. D.

POLYHEDRAL ANGLES.

569. Def. The *opening* of three or more planes which meet at a common point is called a **polyhedral angle.**

570. Def. The common point S is the **vertex** of the angle, and the intersections of the planes SA, SB, etc., are its **edges**; the portions of the planes included between the edges are its **faces**, and the angles formed by the edges are its **face angles.**



571. The *magnitude* of a polyhedral angle depends upon the *relative position* of its faces, and not upon their extent.

572. In a polyhedral angle, every two adjacent edges form a face angle, and every two adjacent faces form a dihedral angle. These face angles and dihedral angles are the *parts* of the polyhedral angle.

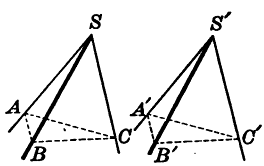
573. Def. A polyhedral angle is **convex**, if every section made by a plane that cuts all its edges is a convex polygon.

574. Def. A polyhedral angle is called **trihedral, tetrahedral,** etc., according as it has *three* faces, *four* faces, etc.

575. Def. A trihedral angle is called **rectangular, bi-rectangular, tri-rectangular,** according as it has *one, two,* or *three* right dihedral angles.

576. Def. A trihedral angle is called **isosceles** if it has two of its face angles equal.

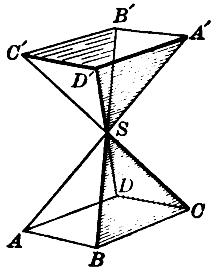
577. Def. Two polyhedral angles can be made to coincide and are **equal** if their corresponding parts are equal and arranged in the **same order.**

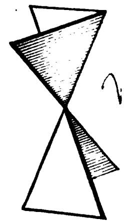


578. A polyhedral angle is designated by its vertex, or by its vertex and all the faces taken in order. Thus the poly-

hedral angle in the margin may be designated by S, or by $S\text{-}ABCD$.

579. If the faces of a polyhedral angle $S\text{-}ABCD$ are produced through the vertex S, another polyhedral angle $S\text{-}A'B'C'D'$ is formed, **symmetrical** with respect to $S\text{-}ABCD$. The face angles ASB, BSC, etc., are equal, respectively, to the face angles $A'SB'$, $B'SC'$, etc. § 93





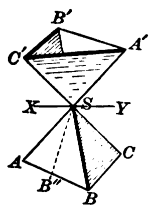
Also the dihedral angles SA, SB, etc., are equal, respectively, to the dihedral angles SA', SB', etc. § 547

(The second figure shows a pair of vertical dihedral angles.)

The edges of $S\text{-}ABCD$ are arranged from **left to right** (counter clockwise) in the order SA, SB, SC, SD, but the edges of $S\text{-}A'B'C'D'$ are arranged from **right to left** (clockwise) in the order SA', SB', SC', SD'; that is, in an order the reverse of the order of the edges in $S\text{-}ABCD$.

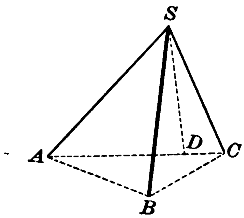
Two symmetrical polyhedral angles, therefore, have all their parts equal, each to each, but arranged in reverse order.

In general, two symmetrical polyhedral angles are not superposable. Thus, if the trihedral angle $S\text{-}A'B'C'$ is made to turn 180° about XY, the bisector of the angle $A'SC$, then SA' will coincide with SC, SC' with SA, and the face $A'SC'$ with ASC; but the dihedral angle SA, and hence the dihedral angle SA', not being equal to SC, the plane $A'SB'$ will not coincide with BSC; and, for a similar reason, the plane $C'SB'$ will not coincide with ASB. Hence, the edge SB' takes some position SB'' not coincident with SB; that is, the trihedral angles are not superposable.



Proposition XXVI. Theorem.

580. *The sum of any two face angles of a trihedral angle is greater than the third face angle.*



In the trihedral angle S-ABC, let the angle ASC be greater than ASB or BSC.

To prove $\angle ASB + \angle BSC$ *greater than* $\angle ASC$.

Proof. In ASC draw SD, making $\angle ASD$ equal to $\angle ASB$.

Through any point D of SD draw ADC in the plane ASC.

Take SB equal to SD.

Pass a plane through the line AC and the point B.

The $\triangle$ ASD and ASB are equal. § 143

For $AS = AS$, $SD = SB$, and $\angle ASD = \angle ASB$.

$\therefore AD = AB$. § 128

In the $\triangle ABC$, $AB + BC > AC$. § 138

But $AB \quad = AD$.

By subtraction, $BC > DC$. Ax. 5

In the $\triangle$ BSC and DSC,

$SC = SC$, and $SB = SD$, but $BC > DC$.

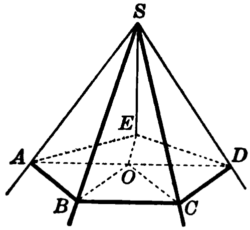
Therefore, $\angle BSC$ is greater than $\angle DSC$. § 155

$\therefore \angle ASB + \angle BSC$ is greater than $\angle ASD + \angle DSC$.

That is, $\angle ASB + \angle BSC$ is greater than $\angle ASC$. Q.E.D.

Proposition XXVII. Theorem.

581. *The sum of the face angles of any convex polyhedral angle is less than four right angles.*



Let S be a convex polyhedral angle, and let all its edges be cut by a plane, making the section ABCDE.

To prove $\angle ASB + \angle BSC$, *etc., less than four rt.* $\angle$s.

Proof. From any point O within the polygon draw OA, OB, OC, OD, OE.

The number of the $\triangle$s having the common vertex O is the same as the number having the common vertex S.

Therefore, the sum of the $\angle$s of all the $\triangle$s having the common vertex S is equal to the sum of the $\angle$s of all the $\triangle$s having the common vertex O.

But in the trihedral $\angle$s formed at A, B, C, etc.,

$\angle SAE + \angle SAB$ is greater than $\angle EAB$,

$\angle SBA + \angle SBC$ is greater than $\angle ABC$, etc. § 580

Hence, the sum of the $\angle$s at the bases of the $\triangle$s whose common vertex is S is greater than the sum of the $\angle$s at the bases of the $\triangle$s whose common vertex is O. Ax. 4

Therefore, the sum of the $\angle$s at the vertex S is less than the sum of the $\angle$s at the vertex O. Ax. 5

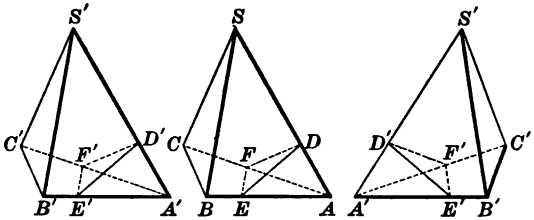
But the sum of the $\angle$s at O is equal to 4 rt. $\angle$s. § 88

Therefore, the sum of the $\angle$s at S is less than 4 rt. $\angle$s.

Q. E. D.

Proposition XXVIII. Theorem.

582. *Two trihedral angles are equal or symmetrical when the three face angles of the one are respectively equal to the three face angles of the other.*



In the trihedral angles S and S′, let the angles ASB, ASC, BSC be equal to the angles A′S′B′, A′S′C′, B′S′C′, respectively.

To prove that S and S′ are equal or symmetrical.

Proof. On the edges of these angles take the six equal distances SA, SB, SC, $S'A'$, $S'B'$, $S'C'$.

Draw AB, BC, AC, $A'B'$, $B'C'$, $A'C'$.

The isosceles ⊿ SAB, SAC, SBC are equal, respectively, to the isosceles ⊿ $S'A'B'$, $S'A'C'$, $S'B'C'$. § 143

$\therefore$ AB, BC, CA are equal, respectively, to $A'B'$, $B'C'$, $C'A'$.

$$\therefore \triangle ABC = \triangle A'B'C'. \qquad \S\ 150$$

At any point D in SA draw DE in the face ASB and DF in the face ASC $\perp$ to SA.

These lines meet AB and AC, respectively,

(*since the ⊿ SAB and SAC are acute, each being one of the equal ⊿ of an isosceles* △).

Draw EF.

On $A'S'$ take $A'D'$ equal to AD.

Draw $D'E'$ in the face $A'S'B'$ and $D'F'$ in the face $A'S'C'$ $\perp$ to $S'A'$, and draw $E'F'$.

The rt. $\triangle$s ADE and $A'D'E'$ are equal. § 142

For $AD = A'D'$, Const.

and $\angle DAE = \angle D'A'E'$. § 128

$\therefore AE = A'E'$, and $DE = D'E'$. § 128

In like manner $AF = A'F'$, and $DF = D'F'$.

$\therefore \triangle AEF = \triangle A'E'F'$. § 143

For $AE = A'E'$, $AF = A'F'$, and $\angle EAF = \angle E'A'F'$. § 128

$\therefore EF = E'F'$. § 128

$\therefore \triangle EDF = \triangle E'D'F'$. § 150

For $ED = E'D'$, $DF = D'F'$, and $EF = E'F'$.

$\therefore \angle EDF = \angle E'D'F'$. § 128

$\therefore$ the angle $B\text{-}AS\text{-}C = B'\text{-}A'S'\text{-}C'$,

(since $\angle$s EDF and $E'D'F'$, the measures of these dihedral $\angle$s, are equal).

In like manner it may be proved that the dihedral angles $A\text{-}BS\text{-}C$ and $A\text{-}CS\text{-}B$ are equal, respectively, to the dihedral angles $A'\text{-}B'S'\text{-}C'$ and $A'\text{-}C'S'\text{-}B'$.

$\therefore S$ and S' are equal or symmetrical. §§ 577, 579

Q.E.D.

This demonstration applies to either of the two figures denoted by $S'\text{-}A'B'C'$, which are symmetrical with respect to each other. If the first of these figures is taken, S and S' are equal. If the second is taken, S and S' are symmetrical.

583. Cor. *If two trihedral angles have three face angles of the one equal to three face angles of the other, then the dihedral angles of the one are respectively equal to the dihedral angles of the other.*

THEOREMS.

Ex. 616. Find the locus of points in space equidistant from two given intersecting lines.

Ex. 617. Find the locus of points in space equidistant from all points in the circumference of a circle.

Ex. 618. Find the locus of points in a plane equidistant from a given point without the plane.

Ex. 619. Find a point at equal distances from four points not all in the same plane.

Ex. 620. Two dihedral angles which have their edges parallel and their faces perpendicular are equal or supplementary.

Ex. 621. The projections on a plane of equal and parallel lines are equal and parallel.

Ex. 622. If two face angles of a trihedral angle are equal, the dihedral angles opposite them are equal.

Ex. 623. The planes that bisect the dihedral angles of a trihedral angle intersect in the same straight line.

Ex. 624. If the face angle ASB of the trihedral angle $S\text{-}ABC$ is bisected by the line SD, the angle CSD is less than half the sum of the angles ASC and BSC.

Ex. 625. An isosceles trihedral angle and its symmetrical trihedral angle are superposable.

Ex. 626. Find the locus of points equidistant from the three edges of a trihedral angle.

Ex. 627. Find the locus of points equidistant from the three faces of a trihedral angle.

Ex. 628. Two trihedral angles are equal when two dihedral angles and the included face angle of the one are equal, respectively, to two dihedral angles and the included face angle of the other and similarly placed.

Ex. 629. Two trihedral angles are equal when two face angles and the included dihedral angle of the one are equal, respectively, to two face angles and the included dihedral angle of the other and similarly placed.

Book VII.

POLYHEDRONS, CYLINDERS, AND CONES.

POLYHEDRONS.

584. Def. A **polyhedron** is a solid bounded by planes.

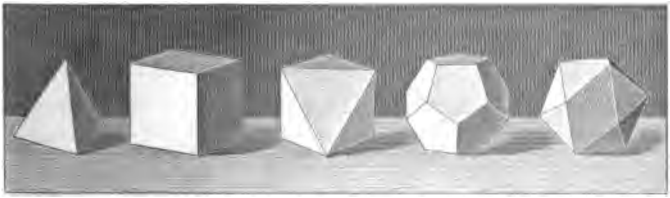
The bounding planes are called the **faces**, the intersections of the faces, the **edges**, and the intersections of the edges, the **vertices**, of the polyhedron.

585. Def. A **diagonal** of a polyhedron is a straight line joining any two vertices not in the same face.

586. Def. A **section** of a polyhedron is the figure formed by its intersection with a plane passing through it.

587. Def. A polyhedron is **convex** if every section is a convex polygon.

Only convex polyhedrons are considered in this work.



Tetrahedron. Hexahedron. Octahedron. Dodecahedron. Icosahedron.

588. Def. A polyhedron of four faces is called a **tetrahedron**; one of six faces, a **hexahedron**; one of eight faces, an **octahedron**; one of twelve faces, a **dodecahedron**; one of twenty faces, an **icosahedron**.

Note. Full lines in the figures of solids represent visible lines, dashed lines represent invisible lines.

PRISMS AND PARALLELOPIPEDS.

589. DEF. A **prism** is a polyhedron of which two faces are equal polygons in parallel planes, and the other faces are parallelograms.

The equal polygons are called the **bases** of the prism, the parallelograms, the **lateral faces,** and the intersections of the lateral faces, the **lateral edges** of the prism.

The sum of the areas of the lateral faces of a prism is called its **lateral area.**

Prism.

590. DEF. The **altitude** of a prism is the perpendicular distance between the planes of its bases.

Right Prism.

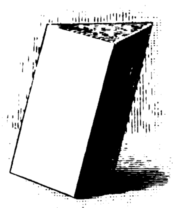
591. DEF. A **right prism** is a prism whose lateral edges are perpendicular to its bases.

592. DEF. A **regular prism** is a right prism whose bases are regular polygons.

593. DEF. An **oblique prism** is a prism whose lateral edges are oblique to its bases.

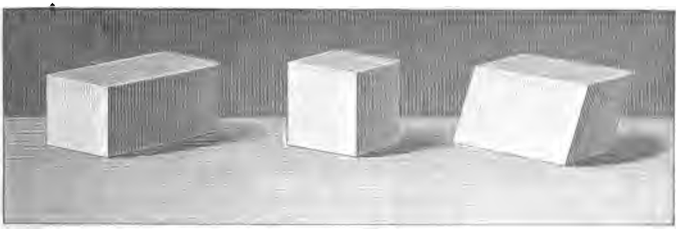
594. The lateral edges of a prism are equal. The lateral edges of a right prism are equal to the altitude.

595. DEF. Prisms are called **triangular, quadrangular,** etc., according as their bases are triangles, quadrilaterals, etc.



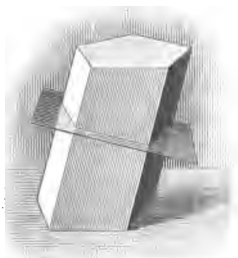
Triangular Prism.

596. DEF. A **parallelopiped** is a prism whose bases are parallelograms.



Rectangular Parallelopiped. **Cube.** **Oblique Parallelopiped.**

597. Def. A **right parallelopiped** is a parallelopiped whose lateral edges are perpendicular to the bases.



Right Section of a Prism.

598. Def. A **rectangular parallelopiped** is a parallelopiped whose six faces are all rectangles.

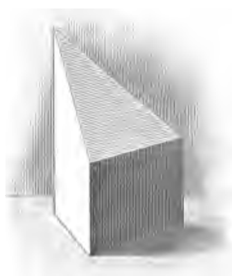
599. Def. A **cube** is a parallelopiped whose six faces are all squares.

600. Def. A cube whose edges are equal to the linear unit is taken as the **unit of volume.**

601. Def. The **volume** of any solid is the number of *units of volume* which it contains.

602. Def. Two solids are **equivalent** if their volumes are equal.

603. Def. A **right section** of a prism is a section made by a plane perpendicular to the lateral edges of the prism.

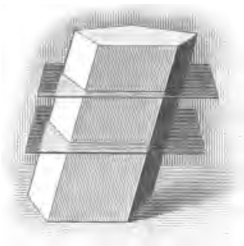


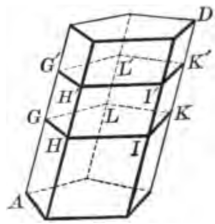
Truncated Prism.

604. Def. A **truncated prism** is the part of a prism included between the base and a section made by a plane oblique to the base.

Proposition I. Theorem.

605. *The sections of a prism made by parallel planes cutting all the lateral edges are equal polygons.*





Let the prism AD be intersected by parallel planes cutting all the lateral edges, making the sections GK, G′K′.

To prove that $GK = G'K'$.

Proof. The sides GH, HI, IK, etc., are parallel, respectively, to the sides $G'H'$, $H'I'$, $I'K'$, etc. § 528

The sides GH, HI, IK, etc., are equal, respectively, to $G'H'$, $H'I'$, $I'K'$, etc. § 180

The $\angle s$ GHI, HIK, etc., are equal, respectively, to $\angle s$ $G'H'I'$, $H'I'K'$, etc. § 534

Therefore, $GK = G'K'$. § 203

Q. E. D.

606. Cor. *Every section of a prism made by a plane parallel to the base is equal to the base; and all right sections of a prism are equal.*

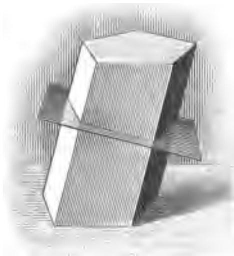
Ex. 630. The diagonals of a parallelopiped bisect one another.

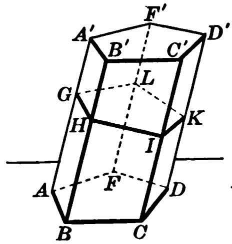
Ex. 631. The lateral faces of a right prism are rectangles.

Ex. 632. Every section of a prism made by a plane parallel to the lateral edges is a parallelogram.

PROPOSITION II. THEOREM.

607. *The lateral area of a prism is equal to the product of a lateral edge by the perimeter of the right section.*





Let GHIKL be a right section of the prism AD′, S its lateral area, E a lateral edge, and P the perimeter of the right section.

To prove that $S = E \times P$.

Proof. $AA' = BB' = CC' = DD' \cdots = E$. § 594

GH is $\perp$ to BB', HI to CC', IK to DD', etc. § 603

$\therefore$ the area of $\square\, AB' = BB' \times GH = E \times GH$, § 400

the area of $\square\, BC' = CC' \times HI = E \times HI$, and so on.

Therefore, S, the sum of these parallelograms, is equal to

$$E(GH + HI + IK + \text{etc.}).$$

But $GH + HI + IK + \text{etc.} = P$.

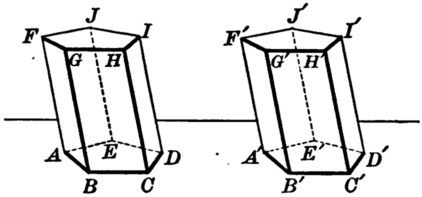
Therefore, $S = E \times P$. Q. E. D.

608. COR. *The lateral area of a right prism is equal to the product of the altitude by the perimeter of the base.*

Ex. 633. Find the lateral area of a right prism, if its altitude is 18 inches and the perimeter of its base 29 inches.

PROPOSITION III. THEOREM.

609. *Two prisms are equal if three faces including a trihedral angle of the one are respectively equal to three faces including a trihedral angle of the other, and are similarly placed.*



In the prisms AI and A′I′, let the faces AD, AG, AJ be respectively equal to A′D′, A′G′, A′J′, and similarly placed.

To prove that $AI = A'I'$.

Proof. The face $\angle s$ BAE, BAF, EAF are equal to the face $\angle s$ $B'A'E'$, $B'A'F'$, $E'A'F'$, respectively. § 203

Therefore, the trihedral angles A and A' are equal. § 582

Apply the trihedral angle A to its equal A'.

Then the face AD coincides with $A'D'$, AG with $A'G'$, and AJ with $A'J'$; and C falls at C', and D at D'.

The lateral edges of the prisms are parallel. § 589

Therefore, CH falls along $C'H'$, and DI along $D'I'$. § 105

Since the points F, G, and J coincide with F', G', and J', each to each, the planes of the upper bases coincide. § 496

Hence, H coincides with H', and I with I'.

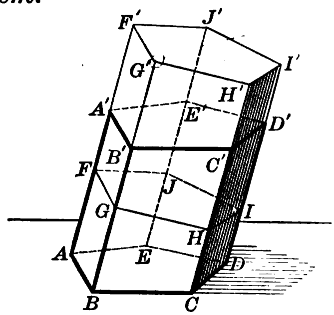
Therefore, the prisms coincide and are equal. Q.E.D.

610. COR. 1. *Two truncated prisms are equal under the hypothesis given in* § 609.

611. COR. 2. *Two right prisms having equal bases and equal altitudes are equal.*

PROPOSITION IV. THEOREM.

612. *An oblique prism is equivalent to a right prism whose base is equal to a right section of the oblique prism, and whose altitude is equal to a lateral edge of the oblique prism.*



Let FI be a right section of the oblique prism AD′, and FI′ a right prism whose lateral edges are equal to the lateral edges of AD′.

To prove that $AD' \approx FI'$.

Proof. If from the equal lateral edges of AD' and FI' we take the lateral edges of FD', which are common to both, the remainders AF and $A'F'$, BG and $B'G'$, etc., are equal. Ax. 3

The upper bases FI and $F'I'$ are equal. § 589

Place AI on $A'I'$ so that FI shall coincide with $F'I'$.

Then FA, GB, etc., coincide with $F'A'$, $G'B'$, etc. § 511

Hence, the faces GA and $G'A'$, HB and $H'B'$, coincide.

But the faces FI and $F'I'$ coincide.

$\therefore$ the truncated prisms AI and $A'I'$ are equal. § 610

Now $$AI + FD' = AD',$$ Ax. 9

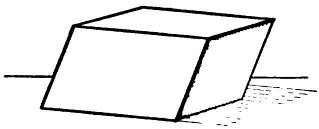
and $$A'I' + FD' = FI'.$$

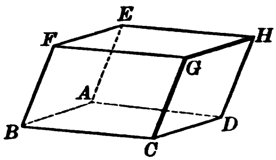
Therefore, $$AD' \approx FI'.$$ Ax. 2

Q. E. D.

Proposition V. Theorem.

613. *The opposite lateral faces of a parallelopiped are equal and parallel.*





Let BH be a parallelopiped with bases BD and FH.

To prove that the opposite faces BG and AH are equal and parallel.

Proof. BC is equal and parallel to AD, §§ 178, 166

and BF is equal and parallel to AE.

$\therefore \angle FBC = \angle EAD$. § 534

$\therefore BG$ is $\parallel$ to AH. § 534

$\therefore BG = AH$. § 185

In like manner it may be proved that the face BE is equal and parallel to CH. Q.E.D.

614. Cor. *Any two opposite faces of a parallelopiped may be taken as bases.*

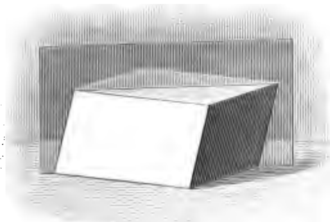
Ex. 634. Find the lateral area of a right prism, if its altitude is 20 inches, and its base is a triangle whose sides are 7 inches, 8 inches, and 9 inches, respectively.

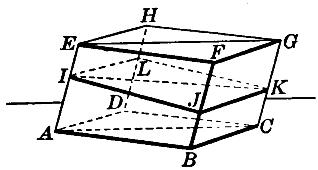
Ex. 635. Find the lateral area of a triangular prism, if its lateral edge is 20 inches, and its right section is a triangle whose sides are 9 inches, 10 inches, and 12 inches, respectively.

Ex. 636. Find the total area of a right prism, if its altitude is 32 inches, and its base is a triangle whose sides are 12 inches, 14 inches, and 16 inches, respectively.

PROPOSITION VI. THEOREM.

615. *The plane passed through two diagonally opposite edges of a parallelopiped divides the parallelopiped into two equivalent triangular prisms.*





Let the plane ACGE pass through the opposite edges AE and CG of the parallelopiped AG.

To prove that the parallelopiped AG is divided into two equivalent triangular prisms ABC-F and ADC-H.

Proof. Let $IJKL$ be a right section of the parallelopiped.

The opposite faces AF and DG, and AH and BG, are parallel and equal. § 613

$\therefore IJ$ is $\parallel$ to LK, and IL to JK. § 528

Therefore, $IJKL$ is a parallelogram. § 166

The intersection IK of the right section with the plane $ACGE$ is the diagonal of the $\square\, IJKL$.

$\therefore \triangle IJK = \triangle ILK.$ § 179

But the prism $ABC\text{-}F$ is equivalent to a right prism whose base is IJK and altitude AE, and the prism $ACD\text{-}H$ is equivalent to a right prism whose base is ILK, and altitude AE. § 612

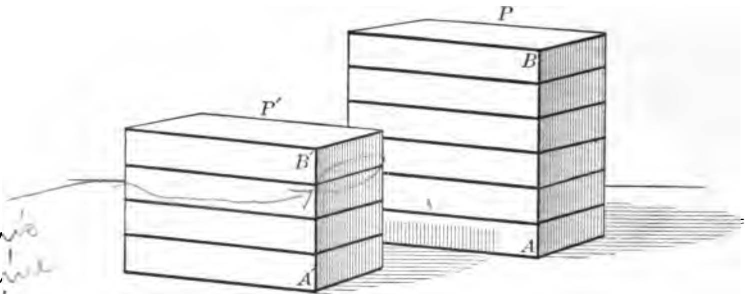
But these two right prisms are equal. § 611

$\therefore ABC\text{-}F \approx ADC\text{-}H.$ Ax. 1

Q. E. D.

Proposition VII. Theorem.

616. *Two rectangular parallelopipeds having equal bases are to each other as their altitudes.*



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Let AB and A′B′ be the altitudes of the two rectangular parallelopipeds P and P′, which have equal bases.

To prove that $P : P' = AB : A'B'$.

Case 1. *When AB and $A'B'$ are commensurable.*

Proof. Find a common measure of AB and $A'B'$.

Apply this common measure to AB and $A'B'$ as a unit of measure.

Suppose this common measure to be contained m times in AB, and n times in $A'B'$.

Then $AB : A'B' = m : n$.

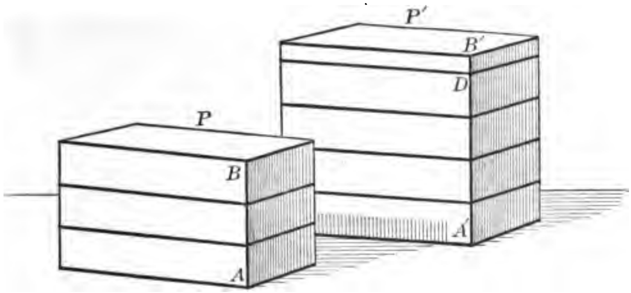
At the several points of division on AB and $A'B'$ pass planes perpendicular to these lines.

The parallelopiped P is divided into m parallelopipeds, and P' into n parallelopipeds, equal each to each. § 611

Therefore, $P : P' = m : n$.

Therefore, $P : P' = AB : A'B'$. Ax. 1

Case 2. *When AB and $A'B'$ are incommensurable.*



Proof. Let AB be divided into any number of equal parts, and let one of these parts be applied to $A'B'$ as a unit of measure as many times as $A'B'$ will contain it.

Since AB and $A'B'$ are incommensurable, a certain number of these parts will extend from A' to a point D, leaving a remainder DB' less than one of the parts.

Through D pass a plane $\perp$ to $A'B'$, and let Q denote the parallelopiped whose base is the same as that of P', and whose altitude is $A'D$.

Then $Q:P = A'D:AB$. Case 1

If the number of parts into which AB is divided is indefinitely increased, the ratio $Q:P$ approaches $P':P$ as a limit, and the ratio $A'D:AB$ approaches $A'B':AB$ as a limit.

The theorem can be proved for this case by the Method of Limits in the manner shown in § 549. Q.E.D.

617. Def. The three edges of a rectangular parallelopiped which meet at a common vertex are called its **dimensions**.

618. Cor. *Two rectangular parallelopipeds which have two dimensions in common are to each other as their third dimensions.*

Proposition VIII. Theorem.

619. *Two rectangular parallelopipeds having equal altitudes are to each other as their bases.*

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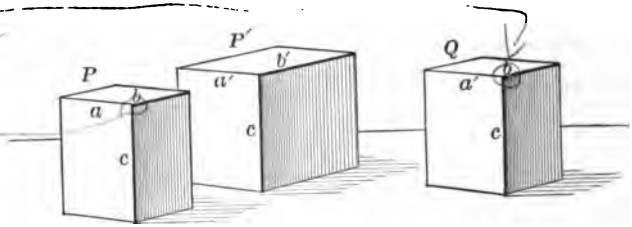


Let a, b, c, and a′, b′, c, be the three dimensions, respectively, of the two rectangular parallelopipeds P and P′.

To prove that $$\frac{P}{P'} = \frac{a \times b}{a' \times b'}.$$

Proof. Let Q be a third rectangular parallelopiped whose dimensions are a', b, and c.

Now Q has the two dimensions b and c in common with P, and the two dimensions a' and c in common with P'.

Then $$\frac{P}{Q} = \frac{a}{a'},$$

and $$\frac{Q}{P'} = \frac{b}{b'}. \qquad \S\ 618$$

The products of the corresponding members of these two equalities give

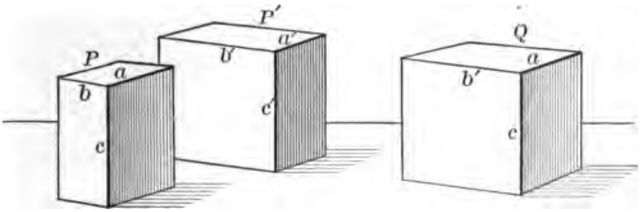
$$\frac{P}{P'} = \frac{a \times b}{a' \times b'}.$$

Q. E. D.

620. Cor. *Two rectangular parallelopipeds which have one dimension in common are to each other as the products of their other two dimensions.*

PROPOSITION IX. THEOREM.

621. *Two rectangular parallelopipeds are to each other as the products of their three dimensions.*



Let a, b, c, and a′, b′, c′, be the three dimensions, respectively, of the two rectangular parallelopipeds P and P′.

To prove that $$\frac{P}{P'} = \frac{a \times b \times c}{a' \times b' \times c'}.$$

Proof. Let Q be a third rectangular parallelopiped whose dimensions are a, b', and c.

Then $$\frac{P}{Q} = \frac{b}{b'},$$ § 618

and $$\frac{Q}{P'} = \frac{a \times c}{a' \times c'}.$$ § 620

The products of the corresponding members of these equalities give

$$\frac{P}{P'} = \frac{a \times b \times c}{a' \times b' \times c'}.$$

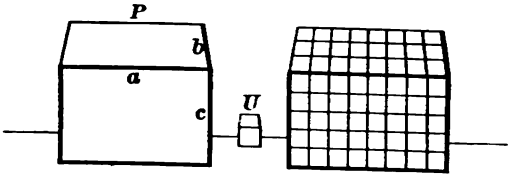
Q. E. D.

Ex. 637. Find the ratio of two rectangular parallelopipeds, if their altitudes are each 6 inches, and their bases 5 inches by 4 inches, and 10 inches by 8 inches, respectively.

Ex. 638. Find the ratio of two rectangular parallelopipeds, if their dimensions are 3, 4, 5, and 9, 8, 10, respectively.

Proposition X. Theorem.

622. *The volume of a rectangular parallelopiped is equal to the product of its three dimensions.*



Let a, b, and c be the three dimensions of the rectangular parallelopiped P, and let the cube U be the unit of volume.

To prove that the volume of $P = a \times b \times c$.

Proof. $$\frac{P}{U} = \frac{a \times b \times c}{1 \times 1 \times 1} = a \times b \times c. \qquad \text{§ 621}$$

Since U is the unit of volume, $\frac{P}{U}$ is the volume of P. § 601

Therefore, the volume of $P = a \times b \times c$. Q. E. D.

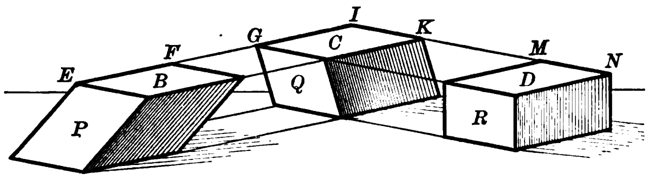
623. Cor. 1. *The volume of a cube is the cube of its edge.*

624. Cor. 2. *The volume of a rectangular parallelopiped is equal to the product of its base by its altitude.*

625. Scholium. When the three dimensions of a rectangular parallelopiped are each exactly divisible by the linear unit, this proposition is rendered evident by dividing the solid into cubes, each equal to the unit of volume. Thus, if the three edges which meet at a common vertex contain the linear unit 3, 5, and 8 times respectively, planes passed through the several points of division of the edges, perpendicular to the edges, will divide the solid into $3 \times 5 \times 8$ cubes, each equal to the unit of volume.

PROPOSITION XI. THEOREM.

626. *The volume of any parallelopiped is equal to the product of its base by its altitude.*



Let P be an oblique parallelopiped no two of whose faces are perpendicular, whose base B is a rhomboid, and whose altitude is H.

To prove that the volume of $P = B \times H$.

Proof. Prolong the edge EF and the edges ∥ to EF, and cut them perpendicularly by two parallel planes whose distance apart GI is equal to EF. We then have the oblique parallelopiped Q whose base C is a rectangle.

Prolong the edge IK and the edges ∥ to IK, and cut them perpendicularly by two planes whose distance apart MN is equal to IK. We then have the rectangular parallelopiped R.

Now $\qquad P \approx Q$, and $Q \approx R$. § 612

$\therefore P \approx R$. Ax. 1

The three solids have a common altitude H. § 530

Also $\qquad B \approx C$; § 401

and $\qquad C = D$. § 186

$\therefore B \approx D$. Ax. 1

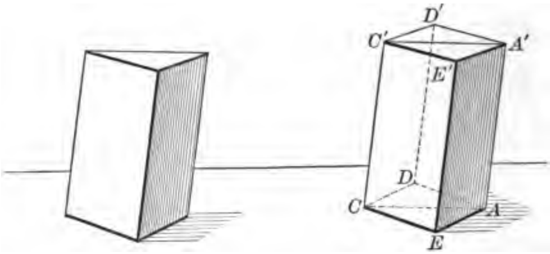
But $\qquad$ the volume of $R = D \times H$. § 624

Putting P for R, and B for D, we have

the volume of $P = B \times H$. Q. E. D.

Proposition XII. Theorem.

627. *The volume of a triangular prism is equal to the product of its base by its altitude.*



Let V denote the volume, B the base, and H the altitude of the triangular prism CEA-E'.

To prove that $V = B \times H$.

Proof. Upon the edges CE, EA, EE', construct the parallelopiped $CEAD\text{-}E'$.

Then $CEA\text{-}E' \approx \frac{1}{2}\, CEAD\text{-}E'$. § 615

Now the volume of $CEAD\text{-}E' = CEAD \times H$. § 626

But $CEAD = 2\,B$. § 179

$\therefore V = \frac{1}{2}(2\,B \times H) = B \times H$. Q. E. D.

Ex. 639. Two triangular prisms are equal if their lateral faces are equal, each to each, and similarly placed.

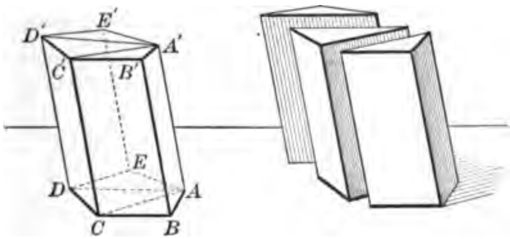
Ex. 640. The square of a diagonal of a rectangular parallelopiped is equal to the sum of the squares of the three dimensions.

Ex. 641. The sum of the squares of the four diagonals of a parallelopiped is equal to the sum of the squares of the twelve edges.

Ex. 642. The volume of a triangular prism is equal to half the product of any lateral face by the perpendicular dropped from the opposite edge on that face.

PROPOSITION XIII. THEOREM.

628. *The volume of any prism is equal to the product of its base by its altitude.*



Let V denote the volume, B the base, and H the altitude of the prism DA′.

To prove that $V = B \times H$.

Proof. Planes passed through the lateral edge AA', and the diagonals AC, AD of the base, divide the given prism into triangular prisms that have the common altitude H.

The volume of each triangular prism is equal to the product of its base by its altitude (§ 627); and hence the sum of the volumes of the triangular prisms is equal to the sum of their bases multiplied by their common altitude.

But the sum of the triangular prisms is equal to the given prism, and the sum of their bases is equal to its base. Ax. 9

Therefore, the volume of the given prism is equal to the product of its base by its altitude.

That is, $V = B \times H$. Q. E. D.

629. COR. *Two prisms are to each other as the products of their bases by their altitudes; prisms having equivalent bases are to each other as their altitudes; prisms having equal altitudes are to each other as their bases; prisms having equivalent bases and equal altitudes are equivalent.*

PROBLEMS OF COMPUTATION.

Ex. 643. If the edge of a cube is 15 inches, find the area of the total surface of the cube.

Ex. 644. If the length of a rectangular parallelopiped is 10 inches, its width 8 inches, and its height 6 inches, find the area of its total surface.

Ex. 645. Find the volume of a right triangular prism, if its height is 14 inches, and the sides of the base are 6, 5, and 5 inches.

Ex. 646. The base of a right prism is a rhombus, one side of which is 10 inches, and the shorter diagonal is 12 inches. The height of the prism is 15 inches. Find the entire surface and the volume.

Ex. 647. Find the volume of a regular prism whose height is 10 feet, if each side of its triangular base is 10 inches.

Ex. 648. How many square feet of lead will be required to line a cistern, open at the top, which is 4 feet 6 inches long, 2 feet 8 inches wide, and contains 42 cubic feet?

Ex. 649. An open cistern 6 feet long and $4\frac{1}{2}$ feet wide holds 108 cubic feet of water. How many square feet of lead will it take to line the sides and bottom?

Ex. 650. An open cistern is made of iron 2 inches thick. The inner dimensions are: length, 4 feet 6 inches; breadth, 3 feet; depth, 2 feet 6 inches. What will the cistern weigh (i) when empty? (ii) when full of water? (The specific gravity of the iron is 7.2.)

Ex. 651. Find the volume of a regular hexagonal prism, if its height is 10 feet, and each side of the hexagon is 10 inches.

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Ex. 652. Find the length of an edge of a cubical vessel that will hold 2 tons of water.

Ex. 653. One edge of a cube is a. Find the surface, the volume, and the length of a diagonal of the cube.

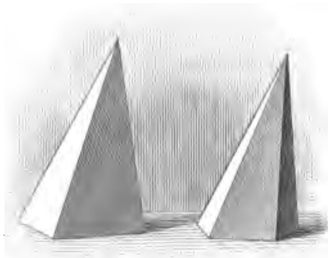
Ex. 654. A diagonal of one of the faces of a cube is a. Find the volume of the cube.

Ex. 655. The three dimensions of a rectangular parallelopiped are a, b, c. Find the area of its surface, its volume, and the length of a diagonal.

Ex. 656. The volume of a parallelopiped is V, and the three dimensions are as $m:n:p$. Find the dimensions.

PYRAMIDS.

630. DEF. A **pyramid** is a polyhedron of which one face, called the **base**, is a polygon of any number of sides and the other faces are triangles having a common vertex.



Pyramids.

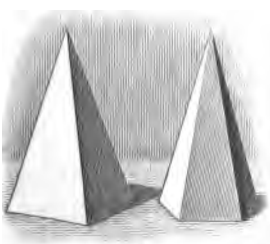
The faces which have a common vertex are called the **lateral faces** of the pyramid, and their common vertex is called the **vertex** of the pyramid.

631. DEF. The intersections of the lateral faces are called the **lateral edges** of the pyramid.

632. DEF. The sum of the areas of the lateral faces is called the **lateral area** of the pyramid.

633. DEF. The **altitude** of a pyramid is the length of the perpendicular let fall from the vertex to the plane of the base.

634. DEF. A pyramid is called **triangular, quadrangular,** etc., according as its base is a triangle, quadrilateral, etc.



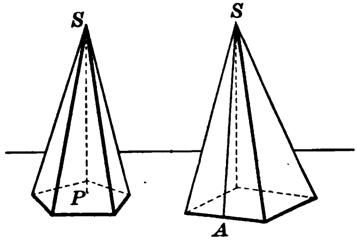
Regular Pyramids.

635. DEF. A triangular pyramid has four triangular faces, and is called a **tetrahedron.** Any one of its faces can be taken for its base.

636. DEF. A pyramid is **regular** if its base is a regular polygon whose centre coincides with the foot of the perpendicular let fall from the vertex to the base.

637. DEF. The perpendicular let fall from the vertex to the base of a regular pyramid is called the **axis** of the pyramid.

The lateral edges of a regular pyramid are equal, for they cut off equal distances from the foot of the perpendicular let fall from the vertex to the base. § 514



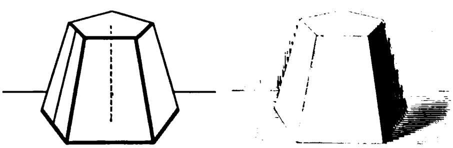
Therefore, the lateral faces of a regular pyramid are equal isosceles triangles. § 150

The altitudes of the lateral faces of a regular pyramid are equal.

638. DEF. The **slant height of a regular pyramid** is the altitude of any one of the lateral faces. It bisects the base of the lateral face in which it is drawn. § 149

639. DEF. A **truncated pyramid** is the portion of a pyramid included between the base and a section made by a plane cutting all the lateral edges.

A **frustum of a pyramid** is the portion of a pyramid included between the base and a section parallel to the base.



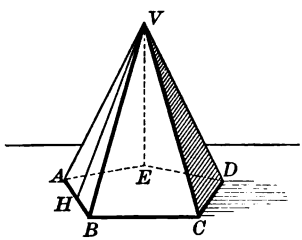
640. DEF. The **altitude of a frustum** is the length of the perpendicular between the planes of its bases.

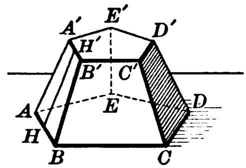
641. DEF. The **lateral faces of a frustum** of a regular pyramid are equal isosceles trapezoids; and the sum of their areas is called the **lateral area of the frustum.**

642. DEF. The **slant height of the frustum** of a regular pyramid is the altitude of one of these trapezoids.

PROPOSITION XIV. THEOREM.

643. *The lateral area of a regular pyramid is equal to half the product of its slant height by the perimeter of its base.*





Let S denote the lateral area of the regular pyramid V-ABCDE, L its slant height, and P the perimeter of its base.

To prove that $S = \frac{1}{2} L \times P.$

Proof. The $\triangle$ VAB, VBC, etc., are equal isosceles $\triangle$. § 637

The area of each $\triangle$ is $\frac{1}{2} L$ multiplied by its base. § 403

$\therefore$ the sum of the areas of these $\triangle$ is $\frac{1}{2} L \times P$.

But the sum of the areas of these $\triangle$ is equal to S, the lateral area of the pyramid.

$$\therefore S = \frac{1}{2} L \times P.$$ Q.E.D.

644. COR. *The lateral area of the frustum of a regular pyramid is equal to half the sum of the perimeters of the bases multiplied by the slant height of the frustum.* § 407

Ex. 657. Find the lateral area of a regular pyramid if the slant height is 16 feet, and the base is a regular hexagon with side 12 feet.

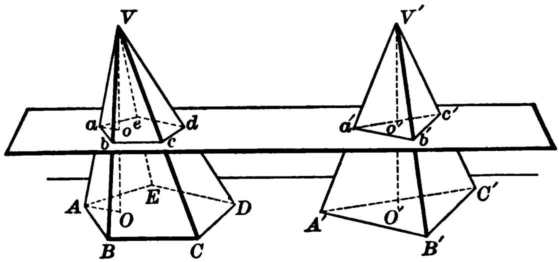
Ex. 658. Find the lateral area of a regular pyramid if the slant height is 8 feet, and the base is a regular pentagon with side 5 feet.

Ex. 659. Find the total surface of a regular pyramid if the slant height is 6 feet, and the base is a square with side 4 feet.

PROPOSITION XV. THEOREM.

645. *If a pyramid is cut by a plane parallel to the base:*

1. *The edges and altitude are divided proportionally.*
2. *The section is a polygon similar to the base.*



Let V-ABCDE be cut by a plane parallel to its base, intersecting the lateral edges in a, b, c, d, e, and the altitude in o.

1. *To prove that* $\dfrac{Va}{VA} = \dfrac{Vb}{VB} \cdots = \dfrac{Vo}{VO}$.

Proof. Since $abcde$ is $\parallel$ to $ABCDE$,

ab is $\parallel$ to AB, bc to $BC \cdots$, and ao is $\parallel$ to AO. § 528

$$\therefore \frac{Va}{VA} = \frac{Vb}{VB} \cdots = \frac{Vo}{VO}. \qquad \text{§ 343}$$

2. *To prove the section abcde similar to the base ABCDE.*

Proof. Since $\triangle\, Vab$ is similar to $\triangle\, VAB$,

$\triangle\, Vbc$ is similar to $\triangle\, VBC$, etc., § 354

$$\frac{ab}{AB} = \left(\frac{Vb}{VB}\right) = \frac{bc}{BC} = \left(\frac{Vc}{VC}\right) = \frac{cd}{CD}, \text{ etc.} \qquad \text{§ 351}$$

$\therefore$ the homologous sides of the polygons are proportional.

Since ab is $\parallel$ to AB, bc to BC, cd to CD, etc., § 528

$\angle abc = \angle ABC$, $\angle bcd = \angle BCD$, etc. § 534

Therefore, the two polygons are mutually equiangular.

Hence, section $abcde$ is similar to the base $ABCDE$. § 351

Q. E. D.

646. Cor. 1. *Any section of a pyramid parallel to the base is to the base as the square of the distance from the vertex is to the square of the altitude of the pyramid.*

For $\frac{Vo}{VO} = \left(\frac{Va}{VA}\right) = \frac{ab}{AB}$. $\therefore \frac{\overline{Vo}^2}{\overline{VO}^2} = \frac{\overline{ab}^2}{\overline{AB}^2}$. § 338

But $\frac{abcde}{ABCDE} = \frac{\overline{ab}^2}{\overline{AB}^2}$. § 412

$\therefore \frac{abcde}{ABCDE} = \frac{\overline{Vo}^2}{\overline{VO}^2}$. Ax. 1

647. Cor. 2. *If two pyramids having equal altitudes are cut by planes parallel to the bases, and at equal distances from the vertices, the sections have the same ratio as the bases.*

For $\frac{abcde}{ABCDE} = \frac{\overline{Vo}^2}{\overline{VO}^2}$,

and $\frac{a'b'c'}{A'B'C'} = \frac{\overline{V'o'}^2}{\overline{V'O'}^2}$. § 646

But $Vo = V'o'$, and $VO = V'O'$. Hyp.

$\therefore abcde : ABCDE = a'b'c' : A'B'C'$.

Whence $abcde : a'b'c' = ABCDE : A'B'C'$. § 330

648. Cor. 3. *If two pyramids have equal altitudes and equivalent bases, sections made by planes parallel to the bases, and at equal distances from the vertices, are equivalent.*

Proposition XVI. Theorem.

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649. *The volume of a triangular pyramid is the limit of the sum of the volumes of a series of inscribed or circumscribed prisms of equal altitude, if the number of prisms is indefinitely increased.*

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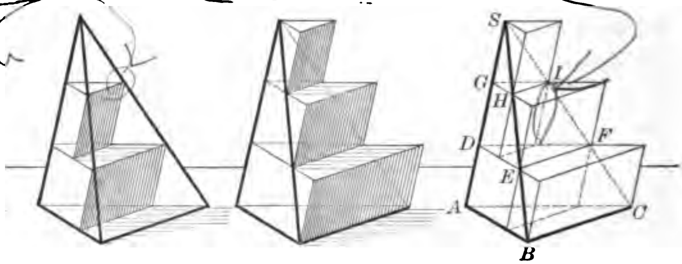


Let S-ABC be a triangular pyramid.

To prove that its volume is the limit of the sum of the volumes of a series of inscribed or circumscribed prisms of equal altitude, if the number of prisms is indefinitely increased.

Proof. Divide the altitude into n equal parts, and denote the length of each part by h.

Through the points of division pass planes parallel to the base, cutting the pyramid in the sections DEF, GHI, etc.

Through EF, HI, etc., pass planes parallel to SA, thus forming a series of prisms which have DEF, GHI, etc., for *upper* bases, and h for altitude.

These prisms may be said to be *inscribed in* the pyramid.

Again, through BC, EF, etc., pass planes parallel to SA, thus forming a series of prisms which have ABC, DEF, etc., for *lower* bases, and h for altitude.

These prisms may be said to be *circumscribed about* the pyramid.

Each inscribed prism is equal to the circumscribed prism directly above it. § 629

The difference, therefore, between the sum of the volumes of the inscribed prisms and the sum of the volumes of the circumscribed prisms is the prism $D\text{-}ABC$.

Denote the volume of the pyramid by P,

the sum of the volumes of the inscribed prisms by V,

the sum of the volumes of the circumscribed prisms, by V',

and the difference between these two sums by d.

Then $$V' - V = d.$$

By increasing n indefinitely, and consequently decreasing h indefinitely, d can be made less than any assigned quantity, however small, but cannot be made zero.

Therefore, $V' - V$ can be made less than any assigned quantity, however small, but cannot be made zero.

Now $V' > P$, and $V < P$. Ax. 8

Therefore, the difference between P and either V' or V is less than $V' - V$.

Therefore, $V' - P$ can be made less than any assigned quantity, however small, but cannot be made zero.

And $P - V$ can be made less than any assigned quantity, however small, but cannot be made zero.

Therefore, P is the common limit of V' and V. § 275

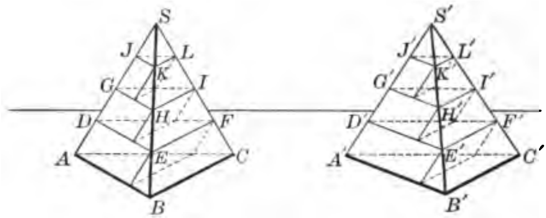
Q. E. D.

Ex. 660. The slant height of a regular pyramid is divided in the ratio 1 : 3 by a plane parallel to the base. Find the ratio of the base to the section.

Ex. 661. The section of a pyramid made by a plane parallel to the base is half as large as the base. Find the ratio of the segments into which the altitude is divided by the plane.

PROPOSITION XVII. THEOREM.

650. *Two triangular pyramids having equivalent bases and equal altitudes are equivalent.*



Let S-ABC and S′-A′B′C′ be two triangular pyramids having equivalent bases situated in the same plane, and equal altitudes.

To prove that $S\text{-}ABC \Leftrightarrow S'\text{-}A'B'C'$.

Proof. Divide the altitude into n equal parts, and through the points of division pass planes parallel to the plane of the bases, forming the sections DEF, GHI, etc., $D'E'F'$, $G'H'I'$, etc.

In the pyramids $S\text{-}ABC$ and $S'\text{-}A'B'C'$ inscribe prisms whose upper bases are the sections DEF, GHI, etc., $D'E'F'$, $G'H'I'$, etc.

The corresponding sections are equivalent. § 648

Therefore, the corresponding prisms are equivalent. § 629

Denote the sum of the volumes of the prisms inscribed in the pyramid $S\text{-}ABC$ by V, and the sum of the volumes of the corresponding prisms inscribed in the pyramid $S'\text{-}A'B'C'$ by V'.

Then $V = V'$. Ax. 2

Now let the number of equal parts into which the altitude is divided be indefinitely increased.

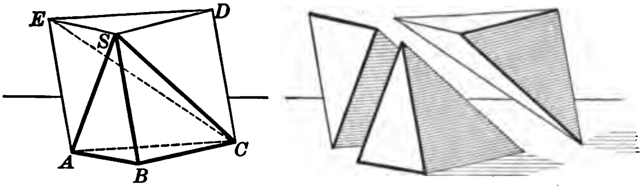
The volumes V and V' are always equal, and approach as limits the pyramids $S\text{-}ABC$ and $S'\text{-}A'B'C'$, respectively. § 649

Hence, $S\text{-}ABC \Leftrightarrow S'\text{-}A'B'C'$. § 284

Q. E. D.

PROPOSITION XVIII. THEOREM.

651. *The volume of a triangular pyramid is equal to one third of the product of its base by its altitude.*



Let V denote the volume, B the base, and H the altitude, of the triangular pyramid S-ABC.

To prove that $V = \frac{1}{3} B \times H.$

Proof. On the base ABC construct a prism ABC-ESD, having its lateral edges equal and parallel to SB.

The prism is composed of the triangular pyramid S-ABC and the quadrangular pyramid S-$ACDE$.

Through SE and SC pass a plane SEC.

This plane divides the quadrangular pyramid into the two triangular pyramids S-ACE and S-DEC, which have the same altitude and equal bases. § 179

$\therefore S\text{-}ACE \approx S\text{-}DEC.$ § 650

The pyramid S-DEC may be regarded as having ESD for its base and C for its vertex, and is, therefore, equivalent to S-ABC. § 650

Hence, the three pyramids into which the prism ABC-ESD is divided are equivalent. Ax. 1

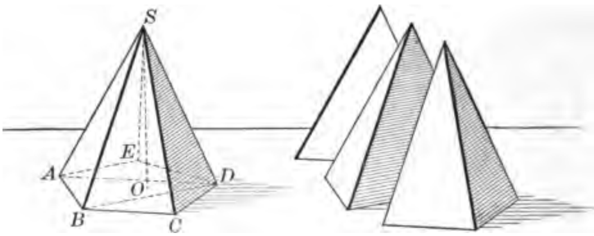
$\therefore$ the pyramid S-ABC is equivalent to one third of the prism.

But the volume of the prism is equal to $B \times H$. § 627

$\therefore V = \frac{1}{3} B \times H.$ Q. E. D.

PROPOSITION XIX. THEOREM.

652. *The volume of any pyramid is equal to one third the product of its base by its altitude.*



Let V denote the volume, B the base, and H the altitude of the pyramid S-ABCDE.

To prove that $V = \frac{1}{3} B \times H$.

Proof. Through the edge SD, and the diagonals of the base DA, DB, pass planes.

These divide the pyramid into triangular pyramids, whose bases are the triangles which compose the base of the pyramid, and whose volumes are together equal to one third the sum of their bases multiplied by their common altitude, H. § 651

That is, $V = \frac{1}{3} B \times H$. Q.E.D.

653. COR. 1. *The volumes of two pyramids are to each other as the products of their bases and altitudes; pyramids of equivalent bases are to each other as their altitudes, and of equal altitudes are to each other as their bases; pyramids having equivalent bases and equal altitudes are equivalent.*

654. COR. 2. *The volume of any polyhedron may be found by dividing it into pyramids, computing their volumes separately, and finding the sum of their volumes.*

PROBLEMS OF COMPUTATION.

Find the volume in cubic feet of a regular pyramid:

Ex. 662. When its base is a square, each side measuring 3 feet 4 inches, and its height is 9 feet.

Ex. 663. When its base is an equilateral triangle, each side measuring 10 feet, and its height is 15 feet.

Ex. 664. When its base is a regular hexagon, each side measuring 4 feet, and its height is 20 feet.

Find the total surface in square feet of a regular pyramid:

Ex. 665. When each side of its square base is 10 feet, and the slant height is 18 feet.

Ex. 666. When each side of its triangular base is 8 feet, and the slant height is 16 feet.

Ex. 667. When each side of its square base is 32 feet, and the perpendicular height is 72 feet.

Find the height in feet of a pyramid when:

Ex. 668. The volume is 26 cubic feet 936 cubic inches, and each side of its square base is 3 feet 6 inches.

Ex. 669. The volume is 20 cubic feet, and the sides of its triangular base are 5 feet, 4 feet, and 3 feet.

Ex. 670. The base edge of a regular pyramid with a square base measures 40 feet, the lateral edge 101 feet. Find its volume in cubic feet.

Ex. 671. Find the volume of a regular pyramid whose slant height is 12 feet, and whose base is an equilateral triangle inscribed in a circle that has a radius of 10 feet.

Ex. 672. Having given the base edge a, and the total surface T, of a regular pyramid with a square base, find the volume V.

Ex. 673. The base edge of a regular pyramid whose base is a square is a, the total surface T. Find the height of the pyramid.

Ex. 674. The eight edges of a regular pyramid with a square base are equal in length, and the total surface is T. Find the length of one edge.

Ex. 675. Find the base edge a of a regular pyramid with a square base, having given the height h and the total surface T.

PROPOSITION XX. THEOREM.

655. *The frustum of a triangular pyramid is equivalent to the sum of three pyramids whose common altitude is the altitude of the frustum and whose bases are the lower base, the upper base, and the mean proportional between the two bases of the frustum.*

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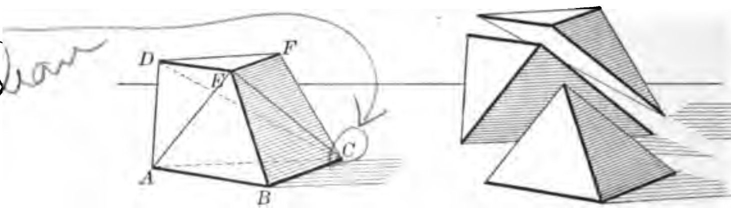


Let B and b denote the lower and upper bases of the frustum ABC-DEF, and H its altitude.

Through the vertices A, E, C and E, D, C pass planes dividing the frustum into three pyramids.

Now the pyramid $E\text{-}ABC$ has for its altitude H, the altitude of the frustum, and for its base B, the lower base of the frustum.

And the pyramid $C\text{-}EDF$ has for its altitude H, the altitude of the frustum, and for its base b, the upper base of the frustum. Hence, it remains

To prove $E\text{-}ADC$ equivalent to a pyramid, having for its altitude H, and for its base $\sqrt{B \times b}$.

Proof. $E\text{-}ABC$ and $E\text{-}ADC$, regarded as having the common vertex C, and their bases in the same plane BD, have a common altitude.

$$\therefore C\text{-}ABE : C\text{-}ADE = \triangle AEB : \triangle AED. \qquad \S\ 653$$

Now the $\triangle$ AEB and AED have a common altitude, the altitude of the trapezoid $ABED$.

$$\therefore \triangle AEB : \triangle AED = AB : DE. \qquad \text{§ 405}$$

$$\therefore C\text{-}ABE : C\text{-}ADE = AB : DE. \qquad \text{Ax. 1}$$

That is, $E\text{-}ABC : E\text{-}ADC = AB : DE.$

In like manner $E\text{-}ADC$ and $E\text{-}DFC$, regarded as having the common vertex E, and their bases in the same plane DC, have a common altitude.

$$\therefore E\text{-}ADC : E\text{-}DFC = \triangle ADC : \triangle DFC. \qquad \text{§ 653}$$

But since the $\triangle$ ADC and DFC have a common altitude, the altitude of the trapezoid $ACFD$,

$$\therefore \triangle ADC : \triangle DFC = AC : DF. \qquad \text{§ 405}$$

$$\therefore E\text{-}ADC : E\text{-}DFC = AC : DF. \qquad \text{Ax. 1}$$

But $\triangle DEF$ is similar to $\triangle ABC$. § 645

$$\therefore AB : DE = AC : DF. \qquad \text{§ 351}$$

$$\therefore E\text{-}ABC : E\text{-}ADC = E\text{-}ADC : E\text{-}DFC. \qquad \text{Ax. 1}$$

Now $E\text{-}ABC = \frac{1}{3} H \times B,$ § 651

and $E\text{-}DFC = C\text{-}EDF = \frac{1}{3} H \times b.$

$$\therefore E\text{-}ADC = \sqrt{\tfrac{1}{3} H \times B \times \tfrac{1}{3} H \times b} = \tfrac{1}{3} H \sqrt{B \times b}.$$

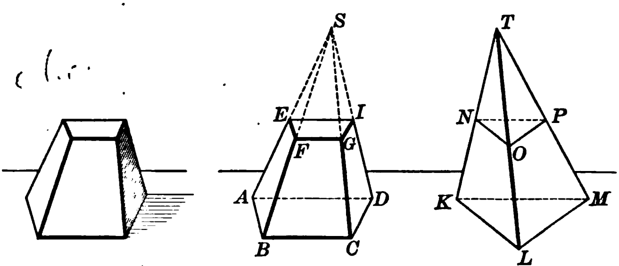
Hence, $E\text{-}ADC$ is equivalent to a pyramid, having for its altitude H, and for its base $\sqrt{B \times b}$. Q.E.D.

656. Cor. *If the volume of a frustum of a triangular pyramid is denoted by* V, *the lower base by* B, *the upper base by* b, *and the altitude by* H,

$$V = \tfrac{1}{3} H \times B + \tfrac{1}{3} H \times b + \tfrac{1}{3} H \times \sqrt{B \times b}$$
$$= \tfrac{1}{3} H \times (B + b + \sqrt{B \times b}).$$

PROPOSITION XXI. THEOREM.

657. *The volume of the frustum of any pyramid is equal to the sum of the volumes of three pyramids whose common altitude is the altitude of the frustum, and whose bases are the lower base, the upper base, and the mean proportional between the bases of the frustum.*



Let B and b denote the lower and upper bases, H the altitude, and V the volume of the frustum ABCD-EFGI.

To prove that $V = \frac{1}{3} H(B + b + \sqrt{B \times b})$.

Proof. Let T-KLM be a triangular pyramid having the same altitude as S-$ABCD$ and its base KLM equivalent to $ABCD$, and lying in the same plane. Then $T\text{-}KLM \approx S\text{-}ABCD$. § 653

Let the plane $EFGI$ cut T-KLM in NOP.

Then $\qquad NOP \approx EFGI.$ § 648

Hence, $\qquad T\text{-}NOP \approx S\text{-}EFGI.$ § 650

Hence, if we take away the upper pyramids, we have left the equivalent frustums NOP-KLM and $EFGI$-$ABCD$.

But the volume of the frustum NOP-KLM is equal to

$$\tfrac{1}{3} H(B + b + \sqrt{B \times b}). \qquad \S\ 656$$

$$\therefore V = \tfrac{1}{3} H(B + b + \sqrt{B \times b}).$$

Q. E. D.

PROPOSITION XXII. THEOREM.

658. *The volumes of two triangular pyramids, having a trihedral angle of the one equal to a trihedral angle of the other, are to each other as the products of the three edges of these trihedral angles.*

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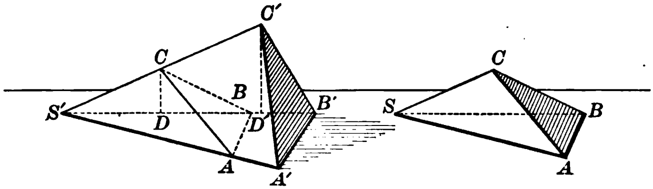


Let V and V′ denote the volumes of the two triangular pyramids S-ABC and S′-A′B′C′, having the trihedral angles S and S′ equal.

To prove that $\dfrac{V}{V'} = \dfrac{SA \times SB \times SC}{S'A' \times S'B' \times S'C'}$.

Proof. Place the pyramid $S\text{-}ABC$ upon $S'\text{-}A'B'C'$ so that the trihedral $\angle\, S$ shall coincide with S'.

Draw CD and $C'D' \perp$ to the plane $S'A'B'$,

and let their plane intersect $S'A'B'$ in $S'DD'$.

The faces $S'AB$ and $S'A'B'$ may be taken as the bases, and CD, $C'D'$ as the altitudes, of the triangular pyramids $C\text{-}S'AB$ and $C'\text{-}S'A'B'$, respectively.

Then $$\frac{V}{V'} = \frac{S'AB \times CD}{S'A'B' \times C'D'} = \frac{S'AB}{S'A'B'} \times \frac{CD}{C'D'}. \qquad \S\ 653$$

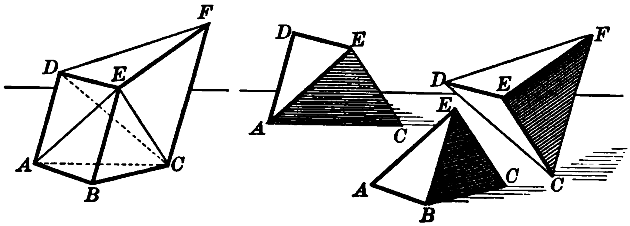
But $$\frac{S'AB}{S'A'B'} = \frac{S'A \times S'B}{S'A' \times S'B'}, \qquad \S\ 410$$

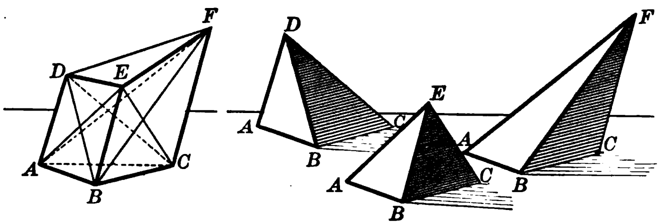
and $$\frac{CD}{C'D'} = \frac{S'C}{S'C'}. \qquad \S\ 351$$

$$\therefore \frac{V}{V'} = \frac{S'A \times S'B \times S'C}{S'A' \times S'B' \times S'C'} = \frac{SA \times SB \times SC}{S'A' \times S'B' \times S'C'}.$$ Q.E.D.

Proposition XXIII. Theorem.

659. *A truncated triangular prism is equivalent to the sum of three pyramids, whose common base is the base of the prism and whose vertices are the three vertices of the inclined section.*





Let ABC-DEF be a truncated triangular prism whose base is ABC, and inclined section DEF.

Pass the planes *AEC* and *DEC*, dividing the truncated prism into the three pyramids *E-ABC*, *E-ACD*, and *E-CDF*.

To prove ABC-DEF equivalent to the sum of the three pyramids, E-ABC, D-ABC, and F-ABC.

Proof. *E-ABC* has the base *ABC* and the vertex *E*.

The pyramid *E-ACD* ⇔ *B-ACD*. § 650

For they have the same base *ACD*, and the same altitude since their vertices *E* and *B* are in the line *EB* ∥ to *ACD*.

But the pyramid *B-ACD* may be regarded as having the base *ABC* and the vertex *D*; that is, as *D-ABC*.

The pyramid *E-CDF* ≎ *B-ACF*. § 650

For their bases *CDF* and *ACF* are equivalent, § 404

since the △ *CDF* and *ACF* have the common base *CF* and equal altitudes, their vertices lying in the line *AD* ∥ to *CF*; and the pyramids have the same altitude, since their vertices *E* and *B* are in the line *EB* ∥ to the plane of their bases.

But the pyramid *B-ACF* may be regarded as having the base *ABC* and the vertex *F*; that is, as *F-ABC*.

Therefore, the truncated triangular prism *ABC-DEF* is equivalent to the sum of the three pyramids *E-ABC*, *D-ABC*, and *F-ABC*. Q.E.D.

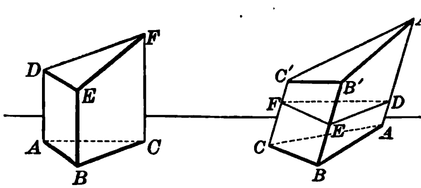


FIG. 1. FIG. 2.

660. COR. 1. *The volume of a truncated right triangular prism is equal to the product of its base by one third the sum of its lateral edges.*

For the lateral edges *DA*, *EB*, *FC* (Fig. 1), being perpendicular to the base *ABC*, are the altitudes of the three pyramids whose sum is equivalent to the truncated prism.

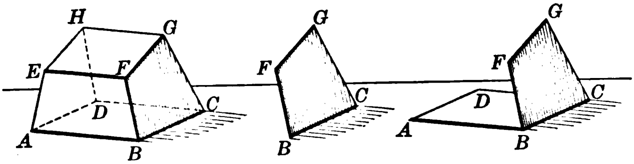
661. COR. 2. *The volume of any truncated triangular prism is equal to the product of its right section by one third the sum of its lateral edges.*

For the right section *DEF* (Fig. 2) divides the truncated prism into two truncated right prisms.

GENERAL THEOREMS OF POLYHEDRONS.

Proposition XXIV. Theorem. (Euler's.)

662. *In any polyhedron the number of edges increased by two is equal to the number of vertices increased by the number of faces.*



Let E denote the number of edges, V the number of vertices, F the number of faces, of the polyhedron AG.

To prove that $E + 2 = V + F.$

Proof. Beginning with one face $BCGF$, we have $E = V$.

Annex a second face $ABCD$, by applying one of its edges to a corresponding edge of the first face, and there is formed a surface of two faces, having *one* edge BC and *two* vertices B and C common to the two faces.

Therefore, for 2 faces $E = V + 1$.

Annex a third face $ABFE$, adjoining each of the first two faces; this face will have *two* edges, AB, BF, and *three* vertices, A, B, F, in common with the surface already formed.

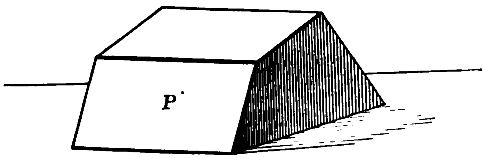
Therefore, for 3 faces $E = V + 2$.
In like manner, for 4 faces $E = V + 3$, and so on.
Therefore, for $(F-1)$ faces $E = V + (F - 2)$.

But $F - 1$ is the number of faces of the polyhedron when only one face is lacking, and the addition of this face will not increase the number of edges or vertices. Hence, for F faces

$$E = V + F - 2, \text{ or } E + 2 = V + F. \qquad \text{Q.E.D.}$$

PROPOSITION XXV. THEOREM.

663. *The sum of the face angles of any polyhedron is equal to four right angles taken as many times, less two, as the polyhedron has vertices.*



Let E denote the number of edges, V the number of vertices, F the number of faces, and S the sum of the face angles, of the polyhedron P.

To prove that $S = (V - 2)\,4 \text{ rt. } \angle s.$

Proof. Since E denotes the number of edges, $2E$ will denote the number of sides of the faces, considered as independent polygons, for each edge is common to two polygons.

If an exterior angle is formed at each vertex of every polygon, the sum of the interior and exterior angles at each vertex is 2 rt. $\angle s$ (§ 86); and since there are $2E$ vertices, the sum of the interior and exterior angles of all the faces is

$$2E \times 2 \text{ rt. } \angle s, \text{ or } E \times 4 \text{ rt. } \angle s.$$

But the sum of the ext. $\angle s$ of each face is 4 rt. $\angle s$. § 207

Therefore, the sum of all the ext. $\angle s$ of F faces is

$$F \times 4 \text{ rt. } \angle s.$$

Therefore, $S = E \times 4 \text{ rt. } \angle s - F \times 4 \text{ rt. } \angle s$

$= (E - F)\,4 \text{ rt. } \angle s.$

But $E + 2 = V + F;$ § 626

that is, $E - F = V - 2.$

Therefore, $S = (V - 2)\,4 \text{ rt. } \angle s.$ Q.E.D.

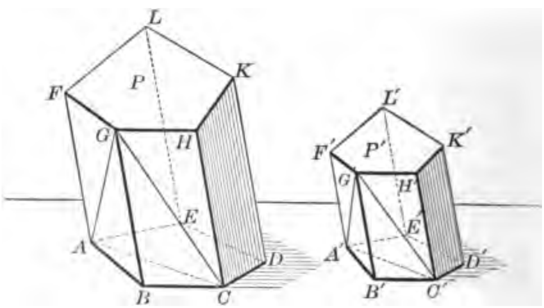
SIMILAR POLYHEDRONS.

664. Def. **Similar polyhedrons** are polyhedrons that have the same number of faces, respectively similar and similarly placed, and have their corresponding polyhedral angles equal.

665. Def. Homologous faces, lines, and angles of similar polyhedrons are faces, lines, and angles similarly situated.

Proposition XXVI. Theorem.

666. *Two similar polyhedrons may be decomposed into the same number of tetrahedrons similar, each to each, and similarly placed.*



Let P and P′ be two similar polyhedrons with G and G′ homologous vertices.

To prove that P and P' can be decomposed into the same number of tetrahedrons, similar each to each, and similarly placed.

Proof. Divide all the faces of P and P', except those which include the angles G and G', into corresponding $\triangle$.

Pass planes through G and the vertices of the $\triangle$ in P; also through G' and the vertices of the $\triangle$ in P'.

Any two corresponding tetrahedrons $G\text{-}ABC$ and $G'\text{-}A'B'C'$ are similar; for they have the faces ABC, GAB, GBC, similar, respectively, to $A'B'C'$, $G'A'B'$, $G'B'C'$; § 365

and the face GAC similar to $G'A'C'$, § 358

since $$\frac{AG}{A'G'} = \left(\frac{AB}{A'B'}\right) = \frac{AC}{A'C'} = \left(\frac{BC}{B'C'}\right) = \frac{GC}{G'C'}.$$ § 351

They also have the corresponding trihedral ∡s equal. § 582

∴ the tetrahedron $G\text{-}ABC$ is similar to $G'\text{-}A'B'C'$. § 664

If $G\text{-}ABC$ and $G'\text{-}A'B'C'$ are removed, the polyhedrons remaining continue similar; for the new faces GAC and $G'A'C'$ have just been proved similar, and the modified faces AGF and $A'G'F'$, CGH and $C'G'H'$, are similar (§ 365); also the modified polyhedral ∡s G and G', A and A', C and C', remain equal each to each, since the corresponding parts taken from them are equal.

The process of removing similar tetrahedrons can be carried on until the polyhedrons are decomposed into the same number of tetrahedrons similar each to each, and similarly placed.

Q. E. D.

667. Cor. 1. *The homologous edges of similar polyhedrons are proportional.* § 351

668. Cor. 2. *Any two homologous lines in two similar polyhedrons have the same ratio as any two homologous edges.* § 351

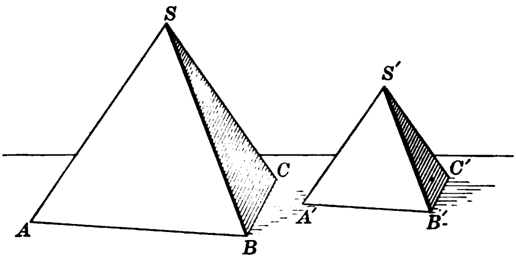
669. Cor. 3. *Two homologous faces of similar polyhedrons are proportional to the squares of two homologous edges.* § 412

670. Cor. 4. *The entire surfaces of two similar polyhedrons are proportional to the squares of two homologous edges.* § 335

Proposition XXVII. Theorem.

671. *The volumes of two similar tetrahedrons are to each other as the cubes of their homologous edges.*

Trim shading



Let V and V′ denote the volumes of the two similar tetrahedrons S-ABC and S′-A′B′C′.

To prove that $$\frac{V}{V'} = \frac{\overline{SB}^3}{\overline{S'B'}^3}.$$

Proof. $$\frac{V}{V'} = \frac{SB \times SC \times SA}{S'B' \times S'C' \times S'A'} \qquad \text{§ 658}$$

$$= \frac{SB}{S'B'} \times \frac{SC}{S'C'} \times \frac{SA}{S'A'}.$$

But $$\frac{SB}{S'B'} = \frac{SC}{S'C'} = \frac{SA}{S'A'}. \qquad \text{§ 667}$$

$$\therefore \frac{V}{V'} = \frac{SB}{S'B'} \times \frac{SB}{S'B'} \times \frac{SB}{S'B'} = \frac{\overline{SB}^3}{\overline{S'B'}^3}.$$

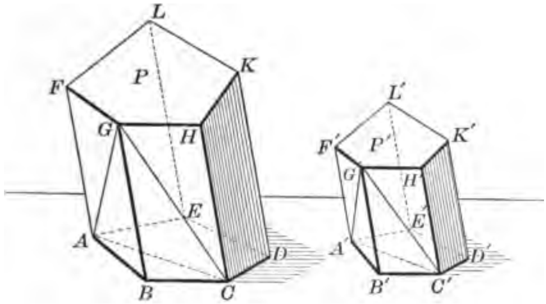
Q. E. D.

Ex. 676. The homologous edges of two similar tetrahedrons are as 6 : 7. Find the ratio of their surfaces and of their volumes.

Ex. 677. If the edge of a tetrahedron is a, find the homologous edge of a similar tetrahedron twice as large.

PROPOSITION XXVIII. THEOREM.

672. *The volumes of two similar polyhedrons are to each other as the cubes of any two homologous edges.*



Let V, V′ denote the volumes, GB, G′B′ any two homologous edges, of the polyhedrons P and P′.

To prove that $V : V' = \overline{GB}^3 : \overline{G'B'}^3.$

Proof. Decompose these polyhedrons into tetrahedrons similar, each to each, and similarly placed. § 666

Denote the volumes of these tetrahedrons by v, v_1, v_2, $\cdots$, v', v_1', v_2', $\cdots$

Then $\dfrac{v}{v'} = \dfrac{\overline{GB}^3}{\overline{G'B'}^3}$, $\dfrac{v_1}{v_1'} = \dfrac{\overline{GB}^3}{\overline{G'B'}^3}$, $\dfrac{v_2}{v_2'} = \dfrac{\overline{GB}^3}{\overline{G'B'}^3}$, and so on. § 671

$$\therefore \frac{v}{v'} = \frac{v_1}{v_1'} = \frac{v_2}{v_2'} = \cdots$$

Whence $$\frac{v + v_1 + v_2 + \cdots}{v' + v_1' + v_2' + \cdots} = \frac{v}{v'} = \frac{\overline{GB}^3}{\overline{G'B'}^3}. \qquad \S\ 335$$

That is, $$\frac{V}{V'} = \frac{\overline{GB}^3}{\overline{G'B'}^3}.$$

Q. E. D.

REGULAR POLYHEDRONS.

673. DEF. A **regular polyhedron** is a polyhedron whose faces are equal regular polygons, and whose polyhedral angles are equal.

PROPOSITION XXIX. PROBLEM.

674. *To determine the number of regular convex polyhedrons possible.*

A convex polyhedral angle must have at least three faces, and the sum of its face angles must be less than 360° (§ 581).

1. Since each angle of an equilateral triangle is 60°, convex polyhedral angles may be formed by combining three, four, or five equilateral triangles. The sum of six such angles is 360°, and hence greater than the sum of the face angles of a convex polyhedral angle. Hence, three regular convex polyhedrons are possible with equilateral triangles for faces.

2. Since each angle of a square is 90°, a convex polyhedral angle may be formed by combining three squares. The sum of four such angles is 360°, and therefore greater than the sum of the face angles of a convex polyhedral angle. Hence, one regular convex polyhedron is possible with squares.

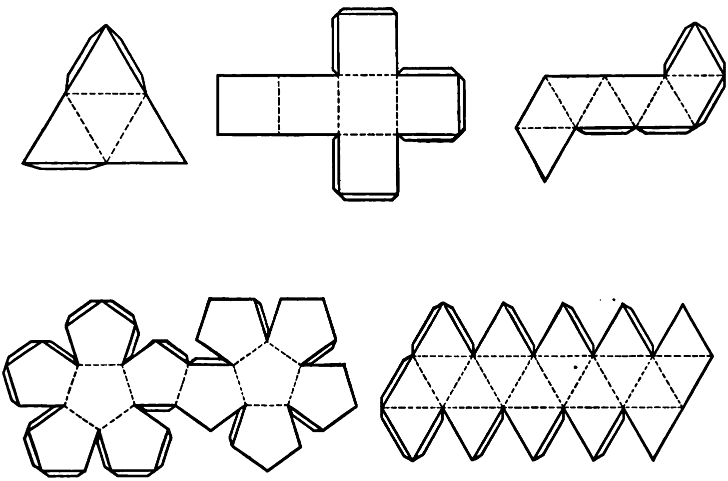
3. Since each angle of a regular pentagon is 108° (§ 206), a convex polyhedral angle may be formed by combining three regular pentagons. The sum of four such angles is 432°, and therefore greater than the sum of the face angles of a convex polyhedral angle. Hence, one regular convex polyhedron is possible with regular pentagons.

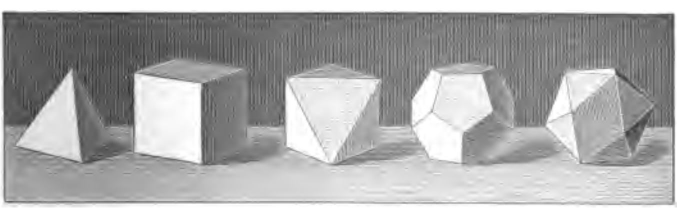
4. The sum of three angles of a regular hexagon is 360°, of a regular heptagon is greater than 360°, etc. Hence, only five regular convex polyhedrons are possible.

The five regular polyhedrons are called the **tetrahedron**, the **hexahedron**, the **octahedron**, the **dodecahedron**, the **icosahedron**.

Q. E. F.

675. The regular polyhedrons may be constructed as follows: Draw the diagrams given below on stiff paper. Cut the ~~through the full lines~~ and fold on the dotted lines. Bring the edges together so as to form the respective polyhedrons, and keep the edges in contact by pasting ~~strips or~~ laps of paper, as shown in the diagrams.



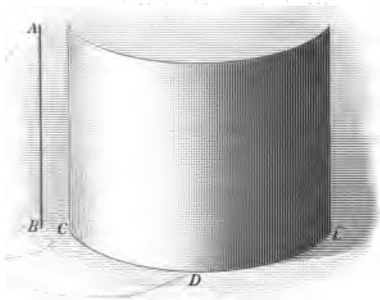


Tetrahedron. Hexahedron. Octahedron. Dodecahedron. Icosahedron.

CYLINDERS.

676. Def. A **cylindrical surface** is a curved surface generated by a straight line, which moves parallel to a fixed straight line and constantly touches a fixed curve not in the plane of the straight line.

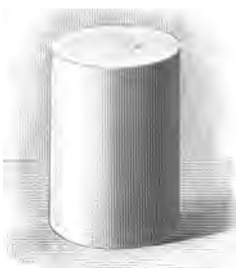
The moving line is called the **generatrix**, and the fixed curve the **directrix**.



Cylindrical Surface.

677. Def. The generatrix in any position is called an **element of the cylindrical surface**.

678. Def. A **cylinder** is a solid bounded by a cylindrical surface and two parallel plane surfaces.

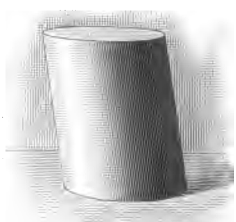


Right Cylinder.

679. Def. The two plane surfaces are called the **bases**, and the cylindrical surface is called the **lateral surface**.

680. Def. The **altitude of a cylinder** is the perpendicular distance between the planes of its bases. The elements of a cylinder are all equal.

681. Def. A cylinder is a **right cylinder** if its elements are perpendicular to its bases; otherwise, an **oblique cylinder**.



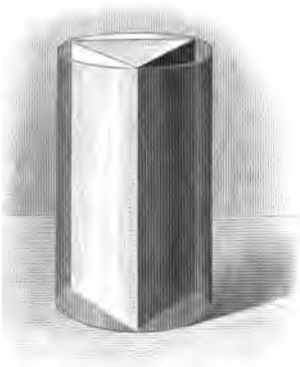
Oblique Cylinder.

682. Def. A **circular cylinder** is a cylinder whose bases are circles.

683. Def. A right circular cylinder is called a **cylinder of revolution** because

?
Insert
See page 340

it may be generated by the revolution of a rectangle about one side as an **axis.**



Inscribed Prism.

684. Def. **Similar cylinders of revolution** are cylinders generated by the revolution of similar rectangles about homologous sides.

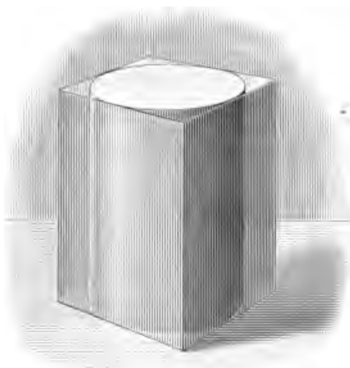
685. Def. A **tangent line to a cylinder** is a straight line, not an element, which touches the lateral surface of the cylinder but does not intersect it.

686. Def. A **tangent plane to a cylinder** is a plane which contains an element of the cylinder but does not cut the surface.

The element contained by the plane is called the **element of contact.**

687. Def. A **prism is inscribed in a cylinder** when its lateral edges are elements of the cylinder and its bases are inscribed in the bases of the cylinder.

688. Def. A **prism is circumscribed about a cylinder** when its lateral edges are parallel to the elements of the cylinder and its bases are circumscribed about the bases of the cylinder.



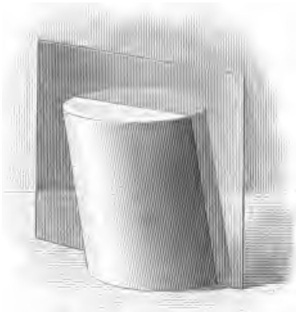
Circumscribed Prism.

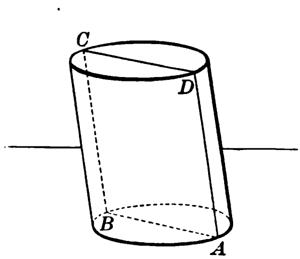
689. Def. A **section of a cylinder** is the figure formed by its intersection with a plane passing through it.

A **right section of a cylinder** is a section made by a plane perpendicular to its elements.

Proposition XXX. Theorem.

690. *Every section of a cylinder made by a plane passing through an element is a parallelogram.*





Let ABCD be a section of the cylinder AC made by a plane passing through the element AD.

To prove that ABCD is a parallelogram.

Proof. Through B draw a line in the plane AC, ∥ to AD.

This line is an element of the cylindrical surface. § 676

Since this line is in both the plane and the cylindrical surface, it must be their intersection and coincide with BC.

Hence, BC coincides with a straight line parallel to AD.

Therefore, BC is a straight line ∥ to AD.

Also AB is a straight line ∥ to CD. § 528

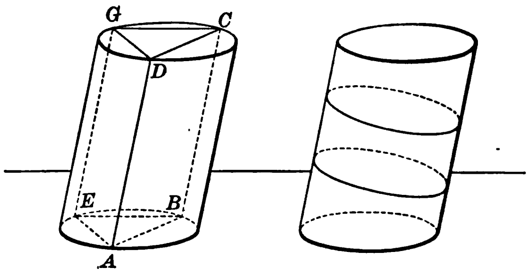
∴ $ABCD$ is a parallelogram. § 166

Q. E. D.

691. Cor. *Every section of a right cylinder made by a plane passing through an element is a rectangle.*

PROPOSITION XXXI. THEOREM.

692. *The bases of a cylinder are equal.*



Let ABE and DCG be the bases of the cylinder AC.

To prove that $ABE = DCG$.

Proof. Let A, B, E be any three points in the perimeter of the lower base, and AD, BC, EG be elements of the surface.

Draw AE, AB, EB, DG, DC, GC.

Then AC, AG, EC are ▱s. § 183

$\therefore AE = DG$, $AB = DC$, and $EB = GC$. § 178

$\therefore \triangle ABE = \triangle DCG$. § 150

Place the lower base on the upper base so that the $\triangle ABE$ shall fall on the $\triangle DCG$. Then A, B, E will fall on D, C, G.

But A, B, E are *any* points in the perimeter of the lower base.

Therefore, *all* points in the perimeter of the lower base will fall on the perimeter of the upper base, and the bases will coincide and be equal. Q. E. D.

693. COR. 1. *Any two parallel sections, cutting all the elements of a cylinder, are equal.*

For these sections are the bases of the included cylinder.

694. COR. 2. *Any section of a cylinder parallel to the base is equal to the base.*

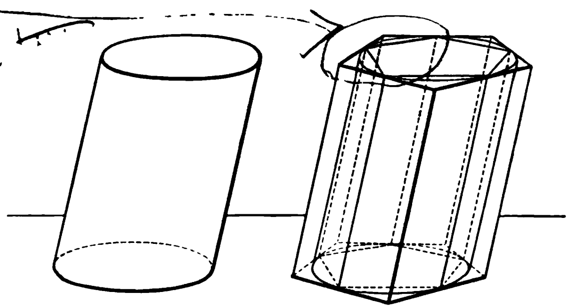
PROPOSITION XXXII. THEOREM.

695. *If a prism whose base is a regular polygon is inscribed in or circumscribed about a circular cylinder, and if the number of sides of the base of the prism is indefinitely increased,*

The volume of the cylinder is the limit of the volume of the prism.

The lateral area of the cylinder is the limit of the lateral area of the prism.

The perimeter of a right section of the cylinder is the limit of the perimeter of a right section of the prism.



Let a prism whose base is a regular polygon be inscribed in a given circular cylinder, and a prism whose base is a regular polygon be circumscribed about the cylinder; and let the number of sides of the base of the prism be indefinitely increased.

To prove that the volume of the cylinder is the limit of the volume of the prism, that the lateral area of the cylinder is the limit of the lateral area of the prism, and that the perimeter of a right section of the cylinder is the limit of the perimeter of a right section of the prism.

Proof. If the bases of the prism and cylinder could be made to coincide exactly, the prism and cylinder would coincide exactly; and their volumes would be equal, their lateral areas would be equal, and the perimeters of their right sections would be equal.

We cannot, however, make the bases of the prism and cylinder coincide exactly, and we cannot, therefore, make the prism and cylinder coincide exactly; but by increasing the number of sides of the base of the prism, we can make the base of the prism come as near coinciding with the base of the cylinder as we choose (§ 454), and consequently make the prism come as near coinciding with the cylinder as we choose.

Therefore, the difference between the volumes of the prism and the cylinder can be made less than any assigned value, however small, but cannot be made zero.

The difference between the lateral areas of the prism and cylinder can be made less than any assigned value, however small, but cannot be made zero.

The difference between the perimeters of the right sections of the prism and cylinder can be made less than any assigned value, however small, but cannot be made zero.

Therefore, the volume of the cylinder is the limit of the volume of the prism. § 275

The lateral area of the cylinder is the limit of the lateral area of the prism. § 275

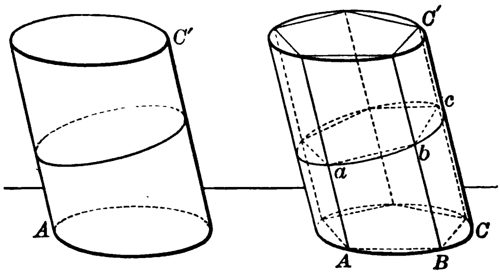
The perimeter of a right section of the cylinder is the limit of the perimeter of a right section of the prism. § 275

Q. E. D.

NOTE. This theorem can be proved true, when the base of the prism is not a regular polygon and the cylinder is not circular; but it is not the province of Elementary Geometry to treat of cylinders whose bases are not circles.

PROPOSITION XXXIII. THEOREM.

696. *The lateral area of a circular cylinder is equal to the product of the perimeter of a right section of the cylinder by an element.*



Let S denote the lateral area, P the perimeter of a right section, and E an element of the cylinder AC′.

To prove that $S = P \times E$.

Proof. Inscribe in the cylinder a prism with its base a regular polygon, and denote its lateral area by S', and the perimeter of its right section by P'.

Then $S' = P' \times E$. § 607

If the number of lateral faces of the inscribed prism is indefinitely increased,

S' approaches S as a limit, § 695

and P' approaches P as a limit. § 695

$\therefore P' \times E$ approaches $P \times E$ as a limit. § 279

But $S' = P' \times E$, always. § 607

$\therefore S = P \times E$. § 284

Q. E. D.

697. COR. 1. *The lateral area of a cylinder of revolution is the product of the circumference of its base by its altitude.*

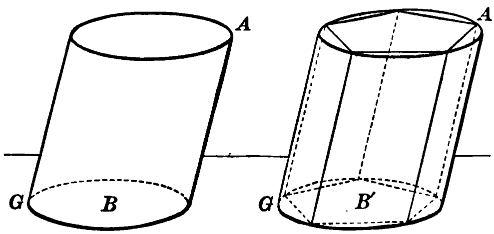
698. Cor. 2. *If* S *denotes the lateral area,* T *the total area,* H *the altitude, and* R *the radius, of a cylinder of revolution,*

$$S = 2\pi R \times H;$$

$$T = 2\pi R \times H + 2\pi R^2 = 2\pi R(H + R).$$

Proposition XXXIV. Theorem.

699. *The volume of a circular cylinder is equal to the product of its base by its altitude.*



Let V denote the volume, B the base, and H the altitude, of the circular cylinder GA.

To prove that $V = B \times H$.

Proof. Inscribe in the cylinder a prism with its base a regular polygon, and denote its volume by V' and its base by B'.

Then $V' = B' \times H$. § 628

If the number of its lateral faces is indefinitely increased,

V' approaches V as a limit, § 695

and B' approaches B as a limit. § 454

$\therefore B' \times H$ approaches $B \times H$ as a limit. § 279

But $V' = B' \times H$, always. § 628

$\therefore V = B \times H$. § 284

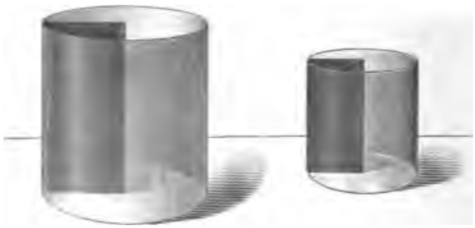
Q. E. D.

700. Cor. *For a cylinder of revolution, with radius* R,

$$V = \pi R^2 \times H.$$

Proposition XXXV. Theorem.

701. *The lateral areas, or the total areas, of similar cylinders of revolution are to each other as the squares of their altitudes, or as the squares of their radii; and their volumes are to each other as the cubes of their altitudes, or as the cubes of their radii.*



Let S, S′ denote the lateral areas, T, T′ the total areas, V, V′ the volumes, H, H′ the altitudes, R, R′ the radii, of two similar cylinders of revolution.

To prove that $S : S' = T : T' = H^2 : H'^2 = R^2 : R'^2$,

and $V : V' = H^3 : H'^3 = R^3 : R'^3$.

Proof. Since the generating rectangles are similar, we have by §§ 351, 335,

$$\frac{H}{H'} = \frac{R}{R'} = \frac{H+R}{H'+R'}.$$

Also we have by §§ 698, 700,

$$\frac{S}{S'} = \frac{2\pi RH}{2\pi R'H'} = \frac{R}{R'} \times \frac{H}{H'} = \frac{R^2}{R'^2} = \frac{H^2}{H'^2}.$$

$$\frac{T}{T'} = \frac{2\pi R(H+R)}{2\pi R'(H'+R')} = \frac{R}{R'}\left(\frac{H+R}{H'+R'}\right) = \frac{R^2}{R'^2} = \frac{H^2}{H'^2}.$$

$$\frac{V}{V'} = \frac{\pi R^2 H}{\pi R'^2 H'} = \frac{R^2}{R'^2} \times \frac{H}{H'} = \frac{R^3}{R'^3} = \frac{H^3}{H'^3}.$$

Q.E.D.

PROBLEMS OF COMPUTATION.

Ex. 678. The diameter of a well is 6 feet, and the water is 7 feet deep. How many gallons of water are there in the well, reckoning $7\frac{1}{2}$ gallons to the cubic foot?

Ex. 679. When a body is placed under water in a right circular cylinder 60 centimeters in diameter, the level of the water rises 40 centimeters. Find the volume of the body.

Ex. 680. How many cubic yards of earth must be removed in constructing a tunnel 100 yards long, whose section is a semicircle with a radius of 18 feet?

Ex. 681. How many square feet of sheet iron are required to make a funnel 18 inches in diameter and 40 feet long?

Ex. 682. Find the radius of a cylindrical pail 14 inches high that will hold exactly 2 cubic feet.

Ex. 683. The height of a cylindrical vessel that will hold 20 liters is equal to the diameter. Find the altitude and the radius.

Ex. 684. If the total surface of a right circular cylinder is T, and the radius of the base is R, find the height of the cylinder.

Ex. 685. If the lateral surface of a right circular cylinder is S, and the volume is V, find the radius of the base and the height.

Ex. 686. If the circumference of the base of a right circular cylinder is C, and the height H, find the volume V.

Ex. 687. Having given the total surface T of a right circular cylinder, in which the height is equal to the diameter of the base, find the volume V.

Ex. 688. If the circumference of the base of a right circular cylinder is C, and the total surface is T, find the volume V.

Ex. 689. If the volume of a right circular cylinder is V, and the altitude is H, find the total surface T.

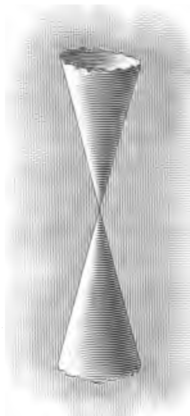
Ex. 690. If V is the volume of a right circular cylinder in which the altitude equals the diameter, find the altitude H, and the total surface T.

Ex. 691. If T is the total surface, and H the altitude of a right circular cylinder, find the radius R, and the volume V.

CONES.

702. Def. A **conical surface** is the surface generated by a moving straight line which constantly touches a fixed curve and passes through a fixed point not in the plane of the curve.

The moving straight line which generates the conical surface is called the **generatrix**, the fixed curve the **directrix**, and the fixed point the **vertex**.

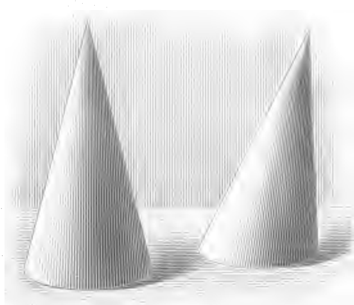


Conical Surface.

703. Def. The generatrix in any position is called an **element of the conical surface**.

If the generatrix is of indefinite length, the surface consists of two portions, one above and the other below the vertex, which are called the **upper and lower nappes**, respectively.

704. Def. If the directrix is a closed curve, the solid bounded by the conical surface and a plane cutting all its elements is called a **cone**.



Cones.

The conical surface is called the **lateral surface of the cone**, and the plane surface is called the **base of the cone**.

The vertex of the conical surface is called the **vertex of the cone**, and the elements of the conical surface are called the **elements of the cone**.

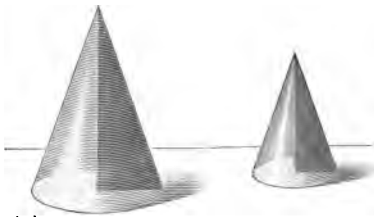
The perpendicular distance from the vertex to the plane of the base is called the **altitude of the cone**.

705. Def. A **circular cone** is a cone whose base is a circle.

The straight line joining the vertex and the centre of the base is called the **axis of the cone.**

If the axis is perpendicular to the base, the cone is called a **right cone.**

If the axis is oblique to the base, the cone is called an **oblique cone.**



Simila Cones of Revolution. Insert simil

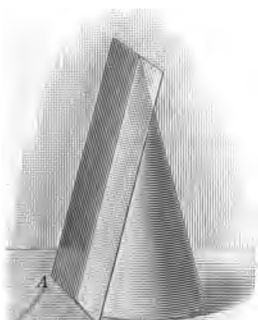
706. Def. A **right circular cone** is a cone whose base is a circle and whose axis is perpendicular to its base.

A right circular cone is called a **cone of revolution,** because it may be generated by the revolution of a right triangle about one of its legs as an axis.

The hypotenuse of the revolving triangle in any position is an element of the surface of the cone, and is called the **slant height** of the cone.

The elements of a cone of revolution are all equal.

707. Def. **Similar cones of revolution** are cones generated by the revolution of similar right triangles about homologous legs.



Tangent Plane.

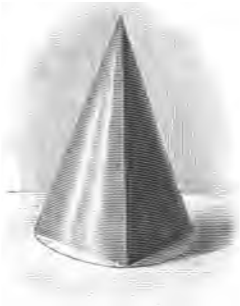
708. Def. A **tangent line to a cone** is a straight line, not an element, which touches the lateral surface of the cone but does not cut it.

709. Def. A **tangent plane to a cone** is a plane which contains an element of the cone but does not cut the surface.

The element contained by the plane is called the **element of contact.**

710. Def. A **pyramid is inscribed in a cone** when its lateral edges are elements of the cone and its base is inscribed in the base of the cone.

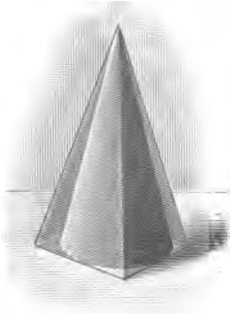
711. Def. A **pyramid is circumscribed about a cone** when its base is circumscribed about the base of the cone and its vertex coincides with the vertex of the cone.



Inscribed Pyramid.

712. Def. A **truncated cone** is the portion of a cone included between the base and a plane cutting all the elements.

A **frustum of a cone** is the portion of a cone included between the base and a plane parallel to the base.



Circumscribed Pyramid.

713. Def. The base of the cone is called the **lower base of the frustum**, and the parallel section is called the **upper base of the frustum.**

714. Def. The **altitude of a frustum of a cone** is the perpendicular distance between the planes of its bases.

715. Def. The **lateral surface of a frustum of a cone** is the portion of the lateral surface of the cone included between the bases of the frustum.

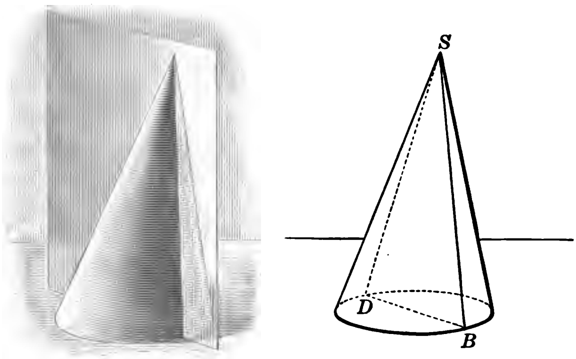
716. Def. The elements of a cone between the bases of a frustum of a cone of revolution are equal, and any one is called the **slant height of the frustum.**

A plane which cuts from the cone a frustum cuts from the inscribed or circumscribed pyramid a frustum.

Frustum of a Cone.

PROPOSITION XXXVI. THEOREM.

717. *Every section of a cone made by a plane passing through its vertex is a triangle.*



Let SBD be a section of the cone SBD made by a plane passing through the vertex S.

To prove that *SBD is a triangle.*

Proof. BD is a straight line. § 506

Draw SB and SD.

These lines are elements of the surface of the cone, and are, therefore, straight lines. § 702

These lines lie in the cutting plane, since their extremities are in the plane. § 492

Hence, SB and SD are the intersections of the conical surface with the cutting plane.

Therefore, the intersections of the conical surface and the plane are straight lines.

Therefore, the section SBD is a triangle. § 117

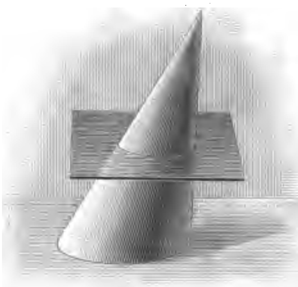
Q. E. D.

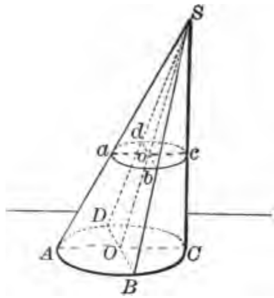
Change [illegible] made

Proposition XXXVII. Theorem.

718. *Every section of a circular cone made by a plane parallel to the base is a circle.*

Change given to [illegible]





a & c A-O-C

Let the section abcd of the circular cone S-ABCD be parallel to the base.

To prove that abcd is a circle.

Proof. Let O be the centre of the base, and let o be the point in which the axis SO pierces the plane of the parallel section.

Through SO and any element SB pass a plane cutting the base in the radius OB, OD, and the section $abcd$ in the straight lines ob, od.

Then ob and od are ∥, respectively, to OB and OD. § 528

Therefore, the $\triangle$ Sob and Sod are similar, respectively, to the $\triangle$ SOB and SOD. § 354

$$\therefore \frac{ob}{OB} = \left(\frac{So}{SO}\right) = \frac{od}{OD}. \qquad \text{§ 351}$$

But $OB = OD$. § 217

$\therefore ob = od$, and $abcd$ is a circle. § 216

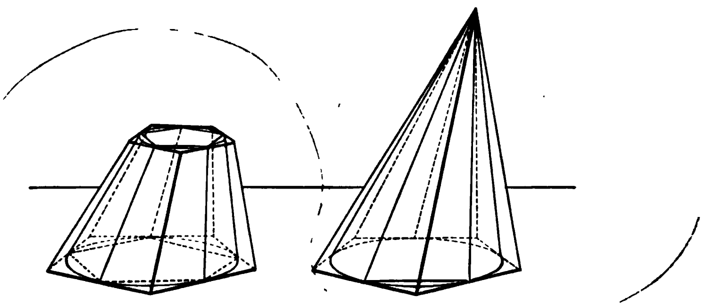
Q. E. D.

719. Cor. *The axis of a circular cone passes through the centre of every section which is parallel to the base.*

? Transpare cuts

Proposition XXXVIII. Theorem.

720. *If a pyramid whose base is a regular polygon is inscribed in or circumscribed about a circular cone, and if the number of sides of the base of the pyramid is indefinitely increased, the volume of the cone is the limit of the volume of the pyramid, and the lateral area of the cone is the limit of the lateral area of the pyramid.*



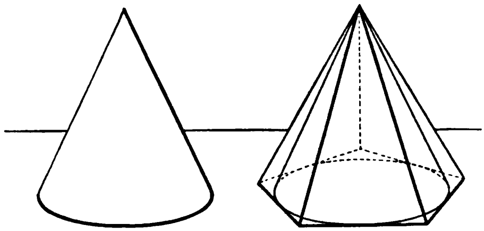
Let a pyramid whose base is a regular polygon be inscribed in a given circular cone, and a pyramid whose base is a regular polygon be circumscribed about the cone, and let the number of sides of the base of the pyramid be indefinitely increased.

The proof is exactly the same as that of Prop. XXXII, if we substitute cone for cylinder and pyramids for prisms.

721. Cor. *The volume of a frustum of a cone is the limit of the volumes of the frustums of the inscribed and circumscribed pyramids, if the number of lateral faces is indefinitely increased, and the lateral area of the frustum of a cone is the limit of the lateral areas of the frustums of the inscribed and circumscribed pyramids.*

PROPOSITION XXXIX. THEOREM.

722. *The lateral area of a cone of revolution is equal to half the product of the slant height by the circumference of the base.*



Let S denote the lateral area, C the circumference of the base, and L the slant height, of the given cone.

To prove that $S = \frac{1}{2} C \times L$.

Proof. Circumscribe about the cone a regular pyramid. Denote the perimeter of its base by P, and its lateral area by S'.

Then $S' = \frac{1}{2} P \times L$. § 643

If the number of the lateral faces of the circumscribed pyramid is indefinitely increased,

S' approaches S as a limit, § 720

and P approaches C as a limit. § 454

$\therefore \frac{1}{2} P \times L$ approaches $\frac{1}{2} C \times L$ as a limit. § 279

But $S' = \frac{1}{2} P \times L$, always. § 643

$\therefore S = \frac{1}{2} C \times L$. § 284

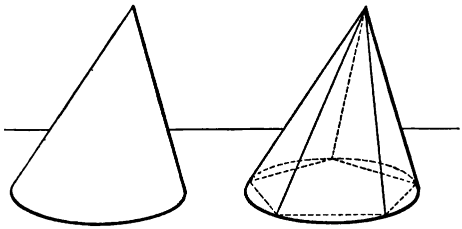
Q.E.D.

723. COR. *If* S *denotes the lateral area,* T *the total area,* H *the altitude,* R *the radius of the base, of a cone of revolution,*

$$S = \tfrac{1}{2}(2\pi R \times L) = \pi RL;$$
$$T = \pi RL + \pi R^2 = \pi R(L + R).$$

PROPOSITION XL. THEOREM.

724. *The volume of a circular cone is equal to one third the product of its base by its altitude.*



Let V denote the volume, B the base, and H the altitude of the given cone.

To prove that $V = \frac{1}{3} B \times H.$

Proof. Inscribe in the cone a pyramid with a regular polygon for its base.

Denote its volume by V', and its base by B'.

Then $V' = \frac{1}{3} B' \times H.$ § 652

If the number of the lateral faces of the inscribed pyramid is indefinitely increased,

V' approaches V as a limit, § 720

and B' approaches B as a limit. § 454

$\therefore \frac{1}{3} B' \times H$ approaches $\frac{1}{3} B \times H$ as a limit. § 279

But $V' = \frac{1}{3} B' \times H$, always. § 652

$\therefore V = \frac{1}{3} B \times H.$ § 284

Q. E. D.

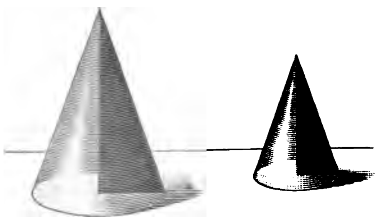
725. COR. *If the cone is a cone of revolution, and* R *is the radius of the base,*

$B = \pi R^2.$ § 463

$\therefore V = \frac{1}{3} \pi R^2 \times H.$

PROPOSITION XLI. THEOREM.

726. *The lateral areas, or the total areas, of two similar cones of revolution are to each other as the squares of their altitudes, as the squares of their radii, or as the squares of their slant heights; and their volumes are to each other as the cubes of their altitudes, as the cubes of their radii, or as the cubes of their slant heights.*



Let S and S′ denote the lateral areas, T and T′ the total areas, V and V′ the volumes, H and H′ the altitudes, R and R′ the radii, L and L′ the slant heights, of two similar cones of revolution.

To prove that $S : S' = T : T' = H^2 : H'^2 = R^2 : R'^2 = L^2 : L'^2$,

and $V : V' = H^3 : H'^3 = R^3 : R'^3 = L^3 : L'^3$.

Proof. $$\frac{H}{H'} = \frac{R}{R'} = \frac{L}{L'} = \frac{L+R}{L'+R'}. \qquad \text{§§ 351, 335}$$

$$\frac{S}{S'} = \frac{\pi RL}{\pi R'L'} = \frac{R}{R'} \times \frac{L}{L'} = \frac{R^2}{R'^2} = \frac{L^2}{L'^2} = \frac{H^2}{H'^2}, \qquad \text{§ 723}$$

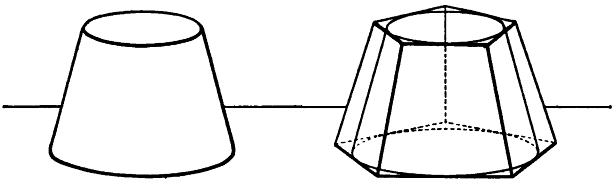
$$\frac{T}{T'} = \frac{\pi R(L+R)}{\pi R'(L'+R')} = \frac{R}{R'} \times \frac{L+R}{L'+R'} = \frac{R^2}{R'^2} = \frac{L^2}{L'^2} = \frac{H^2}{H'^2}.$$

$$\frac{V}{V'} = \frac{\frac{1}{3}\pi R^2 H}{\frac{1}{3}\pi R'^2 H'} = \frac{R^2}{R'^2} \times \frac{H}{H'} = \frac{R^3}{R'^3} = \frac{H^3}{H'^3} = \frac{L^3}{L'^3}. \qquad \text{§ 725}$$

Q. E. D.

PROPOSITION XLII. THEOREM.

727. *The lateral area of a frustum of a cone of revolution is equal to half the sum of the circumferences of its bases multiplied by the slant height.*



Let S denote the lateral area, C and c the circumferences of its bases, R and r their radii, and L the slant height.

To prove that $S = \frac{1}{2}(C + c) \times L.$

Proof. Circumscribe about the frustum of the cone a frustum of a regular pyramid. Denote the lateral area of this frustum by S', the perimeters of its lower and upper bases by P and p, respectively, and its slant height by L.

Then $S' = \frac{1}{2}(P + p) \times L.$ § 644

If the number of lateral faces is indefinitely increased,

S' approaches S as a limit, § 721

and $P + p$ approaches $C + c$ as a limit. §§ 454, 278

$\therefore \frac{1}{2}(P + p)L$ approaches $\frac{1}{2}(C + c)L$ as a limit. § 279

But $S' = \frac{1}{2}(P + p) \times L$, always. § 644

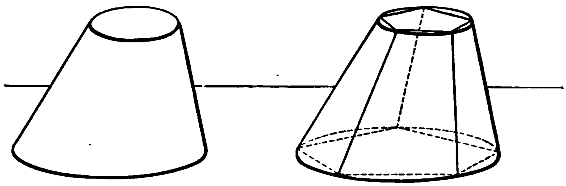
$\therefore S = \frac{1}{2}(C + c) \times L.$ § 284

Q. E. D.

728. COR. *The lateral area of a frustum of a cone of revolution is equal to the circumference of a section equidistant from its bases multiplied by its slant height.*

PROPOSITION XLIII. THEOREM.

729. *The volume of a frustum of a circular cone is equivalent to the sum of the volumes of three cones whose common altitude is the altitude of the frustum and whose bases are the lower base, the upper base, and the mean proportional between the bases of the frustum.*



Let V denote the volume, B the lower base, b the upper base, H the altitude of a frustum of a circular cone.

To prove that $V = \frac{1}{3}H(B + b + \sqrt{B \times b})$.

Proof. Let V' denote the volume, B' and b' the lower and upper bases, and H the altitude, of an inscribed frustum of a pyramid with regular polygons for its bases.

Then $V' = \frac{1}{3}H(B' + b' + \sqrt{B' \times b'})$. § 657

If the number of the lateral faces of the inscribed frustum is indefinitely increased,

V' approaches V as a limit, § 721

B' approaches B as a limit, § 454

and b' approaches b as a limit. § 454

$\therefore B' \times b'$ approaches $B \times b$ as a limit. § 281

$\therefore \sqrt{B' \times b'}$ approaches $\sqrt{B \times b}$ as a limit. § 283

$\therefore B' + b' + \sqrt{B' \times b'}$ approaches

$B + b + \sqrt{B \times b}$ as a limit. § 278

But $$V' = \tfrac{1}{3} H(B' + b' + \sqrt{B' \times b'}), \text{ always.} \qquad \S\ 657$$

$$\therefore V = \tfrac{1}{3} H(B + b + \sqrt{B \times b}). \qquad \S\ 284$$

Q.E.D.

730. Cor. *If the frustum is that of a cone of revolution, and* R *and* r *are the radii of its bases,*

$$B = \pi R^2,\ b = \pi r^2. \qquad \S\ 463$$

$$\therefore \sqrt{B \times b} = \sqrt{\pi R^2 \times \pi r^2} = \pi R r.$$

$$\therefore V = \tfrac{1}{3} \pi H (R^2 + r^2 + Rr).$$

Ex. 692. The radii of the bases of the frustum of a circular cone are 20 inches and 13 inches, respectively. If the altitude of the frustum is 15 inches and is bisected by a plane parallel to the bases, what is the lateral area of each frustum made by the plane?

Ex. 693. The radius of the base of a circular cone is 8 feet, and the altitude is 10 feet. Find the area of its lateral surface, the area of its total surface, and the volume.

Ex. 694. The height of a right circular cone is equal to the diameter of its base. Find the ratio of the area of the base to the area of the lateral surface.

Ex. 695. The slant height of a right circular cone is 2 feet. At what distance from the vertex must the slant height be cut by a plane parallel to the base, in order that the lateral surface may be divided into two equivalent parts?

Ex. 696. What does the volume V of a circular cone become, if the altitude is doubled? If the radius of the base is doubled? If both the altitude and the radius of the base are doubled?

Ex. 697. The slant height L of a right circular cone is equal to the diameter of the base. Find the total surface T.

Ex. 698. If T is the total surface of a right circular cone whose slant height equals the diameter of the base, find the volume V.

Ex. 699. If T is the total surface of a right circular cone, and R is the radius of the base, find the volume V.

Ex. 700. If T is the total surface of a right circular cone, and S is the lateral surface, find the volume V.

THE PRISMATOID FORMULA.

731. Def. A polyhedron is called a **prismatoid** if it has for bases two polygons in parallel planes, and for lateral faces triangles or trapezoids with one side common with one base and the opposite vertex or side common with the other base.

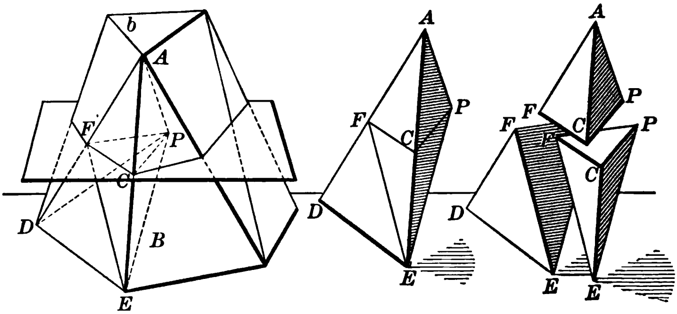
732. Def. The **altitude** of a prismatoid is the perpendicular distance between the planes of its bases.

The **mid-section** of a prismatoid is the section made by a plane parallel to its bases and midway between them.

The mid-section bisects the altitude and all the lateral edges.

Proposition XLIV. Theorem.

733. *The volume of a prismatoid is equal to the product of one sixth of its altitude into the sum of its bases and four times its mid-section.*



Let V denote the volume, B and b the bases, M the mid-section, and H the altitude, of a given prismatoid.

To prove that $V = \frac{1}{6} H (B + b + 4M)$.

Proof. If any lateral face is a trapezoid, divide it into two triangles by a diagonal.

Take any point P in the mid-section and join P to the vertices of the polyhedron and of the mid-section.

Divide the prismatoid into pyramids which have their vertices at P, and for their respective bases the lower base B, the upper base b, and the lateral faces of the prismatoid.

The *lateral* pyramid $P\text{-}ADE$ is composed of three pyramids $P\text{-}AFC$, $P\text{-}FCE$, and $P\text{-}FDE$.

Now $P\text{-}AFC$ may be regarded as having vertex A and base PFC, and $P\text{-}FCE$, as having vertex E and base PFC.

Hence, the volume of $P\text{-}AFC$ is equal to $\frac{1}{6} H \times PFC$,

and the volume of $P\text{-}FCE$ is equal to $\frac{1}{6} H \times PFC$. § 651

The pyramid $P\text{-}FDE$ is equivalent to twice $P\text{-}FCE$.

For they have the same vertex P, and the base FDE is twice the base FCE, since the $\triangle\ FDE$ has its base DE twice the base FC of the $\triangle\ FCE$ (§ 405), and these triangles have the same altitude.

Hence, the volume of $P\text{-}FDE$ is equal to $\frac{2}{6} H \times PFC$.

Therefore, the volume of $P\text{-}ADE$, which is composed of $P\text{-}AFC$, $P\text{-}FCE$, and $P\text{-}FDE$, is equal to $\frac{4}{6} H \times PFC$.

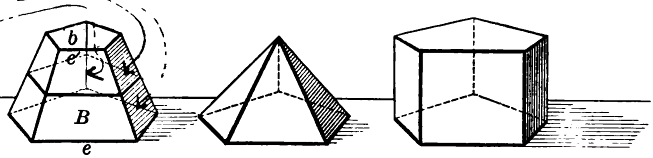
In like manner, the volume of each lateral pyramid is equal to $\frac{4}{6} H \times$ the area of that part of the mid-section which is included within it; and, therefore, the total volume of all these lateral pyramids is equal to $\frac{4}{6} H \times M$.

The volume of the pyramid with base B is $\frac{1}{6} H \times B$,

and the volume of the pyramid with base b is $\frac{1}{6} H \times b$. § 651

Therefore, $V = \frac{1}{6} H (B + b + 4M)$. Q.E.D.

734. *The prismatoid formula may be used for finding the volumes of all the solids of Elementary Geometry:*



The formula for the volume of the frustum of a pyramid is

$$V = \tfrac{1}{3} H(B + b + \sqrt{B \times b}). \qquad \S\ 657$$

But $\tfrac{1}{3} H(B + b + \sqrt{B \times b}) = \tfrac{1}{6} H(B + b + 4M)$. (1)

For if e and e' are corresponding sides of B and b, then $\tfrac{1}{2}(e + e')$ is the corresponding side of M. § 190

Therefore, $\dfrac{e}{\tfrac{1}{2}(e+e')} = \dfrac{\sqrt{B}}{\sqrt{M}}$, and $\dfrac{e'}{\tfrac{1}{2}(e+e')} = \dfrac{\sqrt{b}}{\sqrt{M}}$. § 414

Adding and reducing, $\dfrac{2}{1} = \dfrac{\sqrt{B} + \sqrt{b}}{\sqrt{M}}$.

Therefore, $2\sqrt{M} = \sqrt{B} + \sqrt{b}$.

Squaring, $4M = B + b + 2\sqrt{B \times b}$.

If we put this value of $4M$ in (1), the two members become identically equal.

735. If the base b becomes zero, we have a pyramid, and the prismatoid formula becomes $V = \tfrac{1}{3} H \times B$. § 652

736. If the bases B and b are equal, we have a prism, and the prismatoid formula becomes $V = H \times B$. § 628

NOTE. The Prismatoid Formula is taken by permission from the work on Mensuration by Dr. George Bruce Halsted, Professor of Mathematics in the University of Texas.

Ex. 701. Show that the prismatoid formula can be used for finding the volume of the frustum of a cone, for finding the volume of a cone, for finding the volume of a cylinder.

FRUSTUMS OF PYRAMIDS AND OF CONES.

Ex. 702. How many square feet of tin are required to make a funnel, if the diameters of the top and bottom are 28 inches and 14 inches, respectively, and the height is 24 inches?

Ex. 703. Find the expense, at 60 cents a square foot, of polishing the curved surface of a marble column in the shape of the frustum of a right circular cone whose slant height is 12 feet, and the radii of whose bases are 3 feet 6 inches and 2 feet 4 inches, respectively.

Ex. 704. The slant height of the frustum of a regular pyramid is 20 feet; the sides of its square bases 40 feet and 16 feet. Find the volume.

Ex. 705. If the bases of the frustum of a pyramid are regular hexagons whose sides are 1 foot and 2 feet, respectively, and the volume of the frustum is 12 cubic feet, find the altitude.

Ex. 706. The frustum of a right circular cone 14 feet high has a volume of 924 cubic feet. Find the radii of its bases if their sum is 9 feet.

Ex. 707. From a right circular cone whose slant height is 30 feet, and the circumference of whose base is 10 feet, there is cut off by a plane parallel to the base a cone whose slant height is 6 feet. Find the lateral area and the volume of the frustum.

Ex. 708. Find the difference between the volume of the frustum of a pyramid whose bases are squares, 8 feet and 6 feet, respectively, on a side and the volume of a prism of the same altitude whose base is a section of the frustum parallel to its bases and equidistant from them.

Ex. 709. A Dutch windmill in the shape of the frustum of a right cone is 12 meters high. The outer diameters at the bottom and the top are 16 meters and 12 meters, the inner diameters 12 meters and 10 meters. How many cubic meters of stone were required to build it?

Ex. 710. The chimney of a factory has the shape of a frustum of a regular pyramid. Its height is 180 feet, and its upper and lower bases are squares whose sides are 10 feet and 16 feet, respectively. The flue is throughout a square whose side is 7 feet. How many cubic feet of material does the chimney contain?

Ex. 711. Find the volume V of the frustum of a cone of revolution, having given the slant height L, the height H, and the lateral area S.

EQUIVALENT SOLIDS.

Ex. 712. A cube each edge of which is 12 inches is transformed into a right prism whose base is a rectangle 16 inches long and 12 inches wide. Find the height of the prism, and the difference between its total area and the total area of the cube.

Ex. 713. The dimensions of a rectangular parallelopiped are a, b, c. Find (i) the height of an equivalent right circular cylinder, having a for the radius of its base; (ii) the height of an equivalent right circular cone having a for the radius of its base.

Ex. 714. A regular pyramid 12 feet high is transformed into a regular prism with an equivalent base. Find the height of the prism.

Ex. 715. The diameter of a cylinder is 14 feet, and its height is 8 feet. Find the height of an equivalent right prism, the base of which is a square with a side 4 feet long.

Ex. 716. If one edge of a cube is a, what is the height H of an equivalent right circular cylinder whose radius is R?

Ex. 717. The heights of two equivalent right circular cylinders are in the ratio 4 : 9. If the diameter of the first is 6 feet, what is the diameter of the other?

Ex. 718. A right circular cylinder 6 feet in diameter is equivalent to a right circular cone 7 feet in diameter. If the height of the cone is 8 feet, what is the height of the cylinder?

Ex. 719. The frustum of a regular pyramid 6 feet high has for bases squares 5 feet and 8 feet on a side. Find the height of an equivalent regular pyramid whose base is a square 12 feet on a side.

Ex. 720. The frustum of a cone of revolution is 5 feet high, and the diameters of its bases are 2 feet and 3 feet, respectively. Find the height of an equivalent right circular cylinder whose base is equal in area to the section of the frustum made by a plane parallel to its bases and equidistant from the bases.

Ex. 721. Find the edge of a cube equivalent to a regular tetrahedron whose edge measures 3 inches.

Ex. 722. Find the edge of a cube equivalent to a regular octahedron whose edge measures 3 inches.

SIMILAR SOLIDS.

Ex. 723. The dimensions of a trunk are 4 feet, 3 feet, 2 feet. Find the dimensions of a trunk similar in shape that will hold four times as much.

Ex. 724. By what number must the dimensions of a cylinder be multiplied to obtain a similar cylinder (i) whose surface shall be n times that of the first; (ii) whose volume shall be n times that of the first?

Ex. 725. A pyramid is cut by a plane parallel to the base which passes midmay between the vertex and the plane of the base. Compare the volumes of the entire pyramid and the pyramid cut off.

Ex. 726. The height of a regular hexagonal pyramid is 36 feet, and one side of the base is 6 feet. What are the dimensions of a similar pyramid whose volume is $\frac{1}{20}$ that of the first?

Ex. 727. The length of one of the lateral edges of a pyramid is 4 meters. How far from the vertex will this edge be cut by a plane parallel to the base, which divides the pyramid into two equivalent parts?

Ex. 728. A lateral edge of a pyramid is a. At what distances from the vertex will this edge be cut by two planes parallel to the base that divide the pyramid into three equivalent parts?

Ex. 729. A lateral edge of a pyramid is a. At what distance from the vertex will this edge be cut by a plane parallel to the base that divides the pyramid into two parts which are to each other as $3:4$?

Ex. 730. The volumes of two similar cones are 54 cubic feet and 432 cubic feet. The height of the first is 6 feet; what is the height of the other?

Ex. 731. Two right circular cylinders have their diameters equal to their heights. Their volumes are as $3:4$. Find the ratio of their heights.

Ex. 732. Find the dimensions of a right circular cylinder $\frac{15}{16}$ as large as a similar cylinder whose height is 20 feet, and diameter 10 feet.

Ex. 733. The height of a cone of revolution is H, and the radius of its base is R. Find the dimensions of a similar cone three times as large.

Ex. 734. The height of the frustum of a right cone is $\frac{2}{5}$ the height of the entire cone. Compare the volumes of the frustum and the cone.

Ex. 735. The frustum of a pyramid is 8 feet high, and two homologous edges of its bases are 4 feet and 3 feet, respectively. Compare the volume of the frustum and that of the entire pyramid.

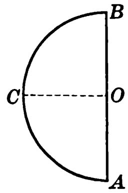
Book VIII.

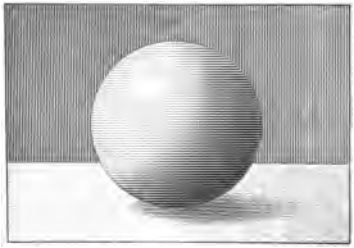
THE SPHERE.

PLANE SECTIONS AND TANGENT PLANES.

737. Def. A **sphere** is a solid bounded by a surface all points of which are equally distant from a point within called the **centre.**

738. A sphere may be generated by the revolution of a semicircle ACB about its diameter AB as an axis.





739. Def. A **radius of a sphere** is a straight line drawn from the centre to the surface.

A **diameter of a sphere** is a straight line passing through the centre and limited by the surface.

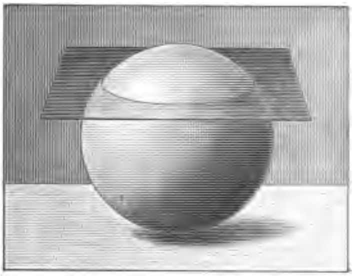
740. *All the radii of a sphere are equal, and all the diameters of a sphere are equal.*

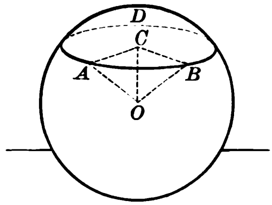
741. Def. A **line or plane is tangent to a sphere** when it has one, and only one, point in common with the surface of the sphere.

742. Def. **Two spheres are tangent to each other** when their surfaces have one, and only one, point in common.

PROPOSITION I. THEOREM.

743. *Every section of a sphere made by a plane is a circle.*





Let O be the centre of the sphere, and ABD any section made by a plane.

To prove that the section ABD is a circle.

Proof. Draw the radii OA, OB, to any two points A, B, in the boundary of the section, and draw $OC \perp$ to the section.

The rt. $\triangle s$ OAC and OBC are equal. § 151

For OC is common, and $OA = OB$. § 740

$\therefore CA = CB$. § 128

But A and B are any two points in the boundary of the section; hence, all points in the boundary are equally distant from C, and the section ABD is a circle. § 216

Q. E. D.

744. COR. 1. *The line joining the centre of a sphere to the centre of a circle of the sphere is perpendicular to the plane of the circle.*

745. COR. 2. *Circles of a sphere made by planes equally distant from the centre are equal.*

For $\overline{AC}^2 = \overline{AO}^2 - \overline{OC}^2$; and AO and OC are the same for all equally distant circles; therefore, AC is the same.

746. Cor. 3. *Of two circles made by planes unequally distant from the centre, the nearer is the greater.*

747. Def. A **great circle of a sphere** is a section made by a plane which passes through the centre of the sphere.

748. Def. A **small circle of a sphere** is a section made by a plane which does not pass through the centre of the sphere.

749. Def. The **axis of a circle of a sphere** is the diameter of the sphere which is perpendicular to the plane of the circle.

The ends of the axis are called the **poles of the circle.**

750. Cor. 1. *Parallel circles have the same axis and the same poles.*

751. Cor. 2. *All great circles of a sphere are equal.*

752. Cor. 3. *Every great circle bisects the sphere.*

For the two parts into which the sphere is divided can be so placed that they will coincide; otherwise there would be points on the surface unequally distant from the centre.

753. Cor. 4. *Two great circles bisect each other.*

For the intersection of their planes passes through the centre, and is, therefore, a diameter of each circle.

754. Cor. 5. *If the planes of two great circles are perpendicular, each circle passes through the poles of the other.*

755. Cor. 6. *Through two given points on the surface of a sphere an arc of a great circle may always be drawn.*

For the two given points together with the centre of the sphere determine the plane of a great circle which passes through the two given points. § 496

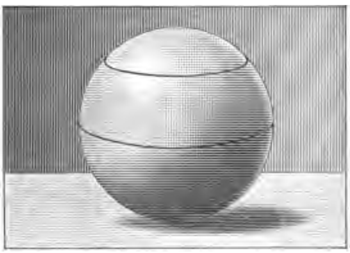
If, however, the two given points are the ends of a diameter, the position of the circle is not determined. § 494

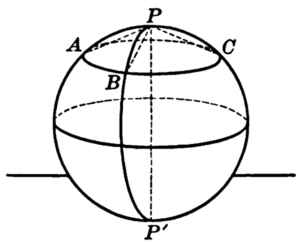
756. Cor. 7. *Through three given points on the surface of a sphere one circle may be drawn, and only one.* § 496

757. Def. The **distance between two points on the surface of a sphere** is the arc of the great circle that joins them.

Proposition II. Theorem.

758. *The distances of all points in the circumference of a circle of a sphere from its poles are equal.*





Let P, P′ be the poles of the circle ABC, and A, B, C, any points in its circumference.

To prove that the great circle arcs PA, PB, PC are equal.

Proof. The straight lines PA, PB, PC are equal. § 514

Therefore, the arcs PA, PB, PC are equal. § 241

In like manner, the great circle arcs $P'A$, $P'B$, $P'C$ may be proved equal. Q.E.D.

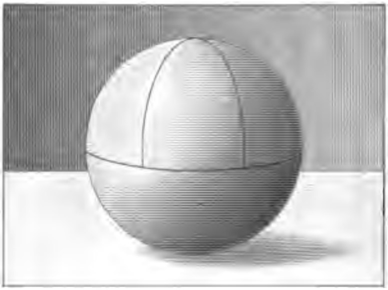
759. Def. The distance on the surface of the sphere from the nearer pole of a small circle to any point in the circumference of the circle is called the **polar distance of the circle.**

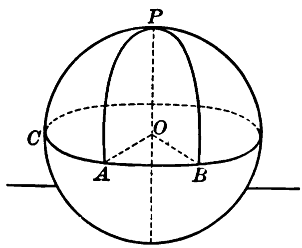
760. Def. The distance on the surface of the sphere of a *great circle from either of its poles* is called the polar distance of the circle.

761. Cor. *The polar distance of a great circle is a quadrant; that is, one fourth the circumference of a great circle.*

PROPOSITION III. THEOREM.

762. *A point on the surface of a sphere, which is at the distance of a quadrant from each of two other points, not the extremities of a diameter, is a pole of the great circle passing through these points.*





Let the distances PA and PB be quadrants, and let ABC be the great circle passing through A and B.

To prove that P is a pole of the great circle ABC.

Proof. The ∠s POA and POB are rt. ∠s. § 288

∴ PO is ⊥ to the plane of the ⊙ ABC. § 507

Hence, P is a pole of the ⊙ ABC. § 749

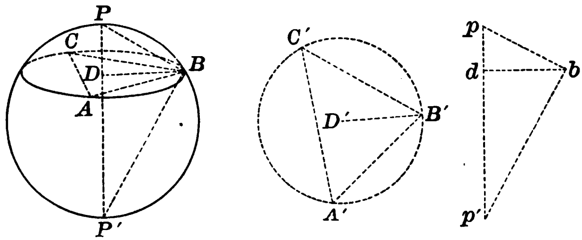
Q. E. D.

763. SCHOLIUM. The above theorem enables us to describe with the compasses an arc of a great circle through two given points of the surface of a sphere. For, if with A and B as centres, and an opening of the compasses equal to the *chord* of a quadrant of a great circle, we describe arcs, these arcs will intersect at a point P. Then, with P as centre, the arc passing through A and B may be described.

In order to make the opening of the compasses equal to the chord of a quadrant of a great circle, the radius of the sphere must be known.

Proposition IV. Problem.

764. *Given a material sphere to find its diameter.*



Let PBP′C represent a material sphere.

It is required to find its diameter.

From any point P of the given surface describe a circumference ABC on the surface.

Then the straight line PB is known.

Take three points A, B, and C in this circumference, and with the compasses measure the chords AB, BC, and CA.

Construct the $\triangle\, A'B'C'$, with sides equal respectively to AB, BC, and CA, and circumscribe a $\odot$ about the $\triangle$.

The radius $D'B'$ of this $\odot$ is equal to the radius of $\odot\, ABC$.

Construct the rt. $\triangle\, bdp$, having the hypotenuse bp equal to BP, and one side bd equal to $B'D'$.

Draw $bp' \perp$ to bp, meeting pd produced in p'.

Then pp' is equal to the diameter of the given sphere.

Proof. Suppose the diameter PP' and the straight line $P'B$ drawn.

The $\triangle$s BDP and bdp are equal. § 151

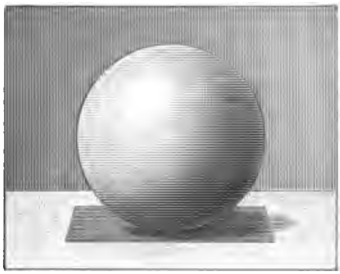
Hence, the $\triangle$s PBP' and pbp' are equal. § 142

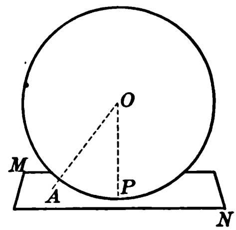
Therefore, $pp' = PP'$. § 128

Q. E. F.

PROPOSITION V. THEOREM.

765. *A plane perpendicular to a radius at its extremity is tangent to the sphere.*





Let O be the centre of a sphere, and MN a plane perpendicular to the radius OP, at its extremity P.

To prove that MN is tangent to the sphere.

Proof. Let A be any point except P in MN. Draw OA.

Then $OP < OA$. § 512

Therefore, the point A is without the sphere. § 737

But A is any point, except P, in the plane MN.

$\therefore$ every point in MN, except P, is without the sphere.

Therefore, MN is tangent to the sphere at P. § 741

Q. E. D.

766. Cor. 1. *A plane tangent to a sphere is perpendicular to the radius drawn to the point of contact.*

767. Cor. 2. *A line tangent to a circle of a sphere lies in the plane tangent to the sphere at the point of contact.* § 508

768. Cor. 3. *A line in a tangent plane drawn through the point of contact is tangent to the sphere at that point.*

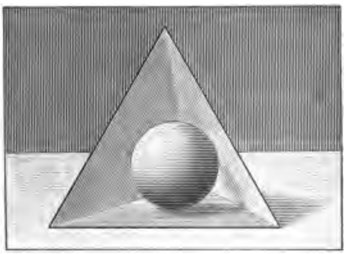
769. Cor. 4. *The plane of two lines tangent to a sphere at the same point is tangent to the sphere at that point.*

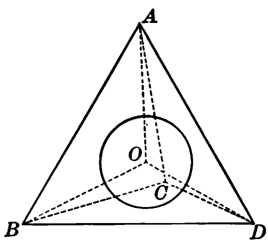
770. DEF. **A sphere is inscribed in a polyhedron** when all the faces of the polyhedron are tangent to the sphere.

771. DEF. **A sphere is circumscribed about a polyhedron** when all the vertices of the polyhedron lie in the surface of the sphere.

PROPOSITION VI. THEOREM.

772. *A sphere may be inscribed in any given tetrahedron.*





Let A-BCD be the given tetrahedron.

To prove that a sphere may be inscribed in A-BCD.

Proof. Bisect the dihedral $\angle s$ at the edges AB, BC, and AC by the planes OAB, OBC, and OAC, respectively.

Every point in the plane OAB is equally distant from the faces ABC and ABD. § 559

For a like reason, every point in the plane OBC is equally distant from the faces ABC and DBC; and every point in the plane OAC is equally distant from the faces ABC and ADC.

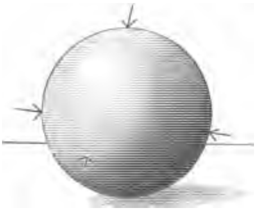
Therefore, O, the common intersection of these three planes, is equally distant from the four faces of the tetrahedron, and is the centre of the sphere inscribed in the tetrahedron. § 770

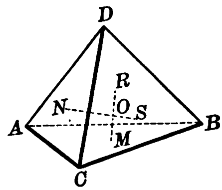
Q. E. D.

773. COR. *The six planes which bisect the six dihedral angles of a tetrahedron intersect in the same point.*

Proposition VII. Theorem.

774. *A sphere may be circumscribed about any given tetrahedron.*





Let D-ABC be the given tetrahedron.

To prove that a sphere may be circumscribed about D-ABC.

Proof. Let M, N, respectively, be the centres of the circles circumscribed about the faces ABC, ACD.

Let MR be $\perp$ to the face ABC, NS $\perp$ to the face ACD.

Then MR is the locus of points equidistant from A, B, C, and NS is the locus of points equidistant from A, C, D. § 516

Therefore, MR and NS lie in the same plane, the plane $\perp$ to AC at its middle point. § 517

Also MR and NS, being $\perp$ to planes which are not $\parallel$, cannot be $\parallel$, and must therefore meet at some point O.

$\therefore$ O is equidistant from A, B, C, and D.

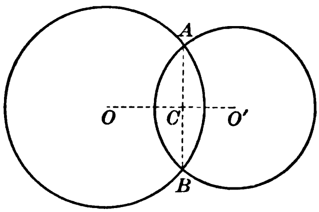
Therefore, a spherical surface whose centre is O, and radius OA, will pass through the points A, B, C, and D. Q.E.D.

775. Cor. 1. *The four perpendiculars erected at the centres of the faces of a tetrahedron meet at the same point.*

776. Cor. 2. *The six planes perpendicular to the edges of a tetrahedron at their middle points intersect at the same point.*

PROPOSITION VIII. THEOREM.

777. *The intersection of two spherical surfaces is the circumference of a circle whose plane is perpendicular to the line joining the centres of the surfaces and whose centre is in that line.*



Let O, O′ be the centres of the spherical surfaces, and let a plane passing through O, O′ cut the spheres in great circles whose circumferences intersect in the points A and B.

To prove that the spherical surfaces intersect in the circumference of a circle whose plane is perpendicular to OO', and whose centre is the point C where AB meets OO'.

Proof. The common chord AB is $\perp$ to OO' and bisected at C. § 264

If the plane of the two great circles is revolved about OO' as an axis, their circumferences will generate the two spherical surfaces, and the point A will describe the line of intersection of the surfaces.

But during the revolution AC will remain constant in length and $\perp$ to OO'.

Therefore, the line of intersection described by the point A will be the circumference of a circle whose centre is C and whose plane is $\perp$ to OO'. § 508

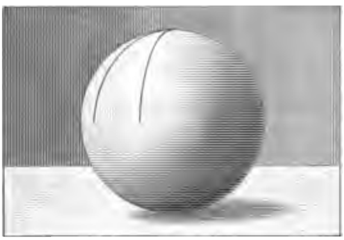
Q. E. D.

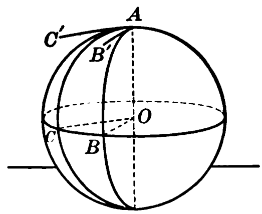
FIGURES ON THE SURFACE OF A SPHERE.

778. Def. The **angle of two curves** passing through the same point is the angle formed by two tangents to the curves at that point. The angle formed by the intersection of two arcs of great circles of a sphere is called a **spherical angle.**

Proposition IX. Theorem.

779. *A spherical angle is measured by the arc of the great circle described from its vertex as a pole and included between its sides (produced if necessary).*





Let AB, AC be arcs of great circles intersecting at A; AB′ and AC′, the tangents to these arcs at A; BC the arc of the great circle described from A as a pole and included between AB and AC.

To prove that the spherical $\angle BAC$ *is measured by arc* BC.

Proof. In the plane AOB, AB' is $\perp$ to AO, § 254

and OB is $\perp$ to AO. § 288

$\therefore AB'$ is $\parallel$ to OB. § 104

Similarly, AC' is $\parallel$ to OC.

$\therefore \angle B'AC' = \angle BOC$. § 534

But $\angle BOC$ is measured by arc BC. § 288

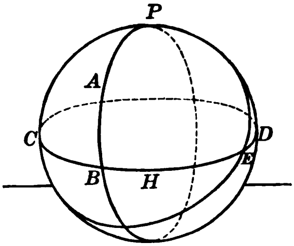
$\therefore \angle B'AC'$ is measured by arc BC.

$\therefore \angle BAC$ is measured by arc BC. Q.E.D.

780. Cor. *A spherical angle has the same measure as the dihedral angle formed by the planes of the two circles.*

Proposition X. Problem.

781. *To describe an arc of a great circle through a given point perpendicular to a given arc of a great circle.*



Let A be a point on the surface of a sphere, CHD an arc of a great circle, P its pole.

From A as a pole describe an arc of a great circle cutting CHD at E.

From E as a pole describe the arc AB through A.

Then AB is the arc required.

Proof. The arc AB is the arc of a great circle, and E is its pole by construction. § 762

The point E is at the distance of a quadrant from P. § 761

Therefore, the arc AB produced will pass through P.

Since the spherical $\angle PBE$ is measured by the arc PE of a great circle (§ 779), the $\angle ABE$ is a right angle.

Therefore, the arc AB is $\perp$ to the arc CHD. Q.E.F.

Ex. 736. Every point in a great circle which bisects a given arc of a great circle at right angles is equidistant from the extremities of the given arc.

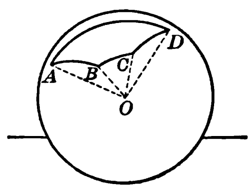
782. Def. A **spherical polygon** is a portion of the surface of a sphere bounded by three or more arcs of great circles.

The bounding arcs are the **sides** of the polygon; the angles between the sides are the **angles** of the polygon; the points of intersection of the sides are the **vertices** of the polygon.

The values of the sides of a spherical polygon are usually expressed in degrees, minutes, and seconds.

783. The planes of the sides of a spherical polygon form a polyhedral angle whose vertex is the centre of the sphere, whose face angles are measured by the sides of the polygon, and whose dihedral angles have the same numerical measure as the angles of the polygon.

Thus, the planes of the sides of the polygon *ABCD* form the polyhedral angle *O-ABCD*. The face angles *AOB*, *BOC*, etc., are measured by the sides *AB*, *BC*, etc., of the polygon. The dihedral angle whose edge is *OA* has the same measure as the spherical angle *BAD*, etc. Hence,



784. *From any property of polyhedral angles we may infer an analogous property of spherical polygons; and conversely.*

785. Def. A **spherical polygon is convex** if the corresponding polyhedral angle is convex (§ 573). Every spherical polygon is assumed to be convex unless otherwise stated.

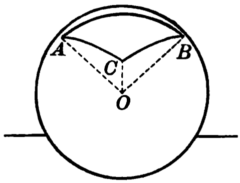
786. Def. A **diagonal of a spherical polygon** is an arc of a great circle connecting any two vertices which are not adjacent.

787. Def. A **spherical triangle** is a spherical polygon of three sides; like a plane triangle, it may be *right*, *obtuse*, or *acute*; *equilateral*, *isosceles*, or *scalene*.

788. Def. **Two spherical polygons are equal** if they can be applied, the one to the other, so as to coincide.

PROPOSITION XI. THEOREM.

789. *Each side of a spherical triangle is less than the sum of the other two sides.*



Let ABC be a spherical triangle, AB the longest side.

To prove that $AB < AC + BC.$

Proof. In the corresponding trihedral angle $O\text{-}ABC$,

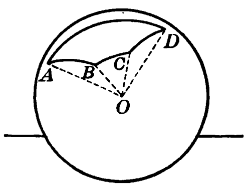
$\angle AOB$ is less than $\angle AOC + \angle BOC$. § 580

$\therefore AB < AC + BC.$ § 783

Q. E. D.

PROPOSITION XII. THEOREM.

790. *The sum of the sides of a spherical polygon is less than 360°.*



Let ABCD be a spherical polygon.

To prove that $AB + BC + CD + DA < 360°.$

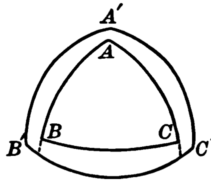
Proof. In the corresponding polyhedral angle $O\text{-}ABCD$, the sum of all the face angles is less than 360°. § 581

$\therefore AB + BC + CD + DA < 360°.$ § 783

Q. E. D.

791. DEF. If, from the vertices of a spherical triangle as poles, arcs of great circles are described, another spherical triangle is formed, called the **polar triangle** of the first.

Thus, if A is the pole of the arc of the great circle $B'C'$, B of $A'C'$, C of $A'B'$, $A'B'C'$ is the polar triangle of ABC.

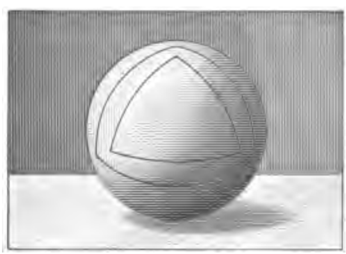


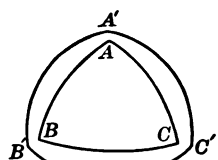
If, with A, B, C as poles, entire great circles are described, these circles divide the surface of the sphere into *eight* spherical triangles.

Of these eight triangles, that one is the polar of ABC whose vertex A', corresponding to A, lies on the same side of BC as the vertex A; and similarly for the other vertices.

PROPOSITION XIII. THEOREM.

792. *If $A'B'C'$ is the polar triangle of ABC, then, reciprocally, ABC is the polar triangle of $A'B'C'$.*





Let A′B′C′ be the polar triangle of ABC.

To prove that ABC is the polar triangle of $A'B'C'$.

Proof. A is the pole of $B'C'$, and C is the pole of $A'B'$. § 791

$\therefore$ B' is at a quadrant's distance from A and C. § 761

$\therefore$ B' is the pole of the arc AC. § 762

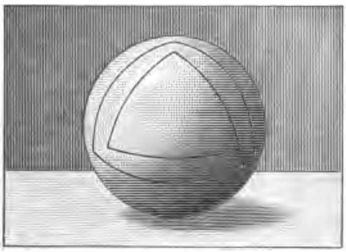
Similarly, A' is the pole of BC, and C' the pole of AB.

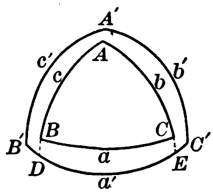
$\therefore$ ABC is the polar triangle of $A'B'C'$. § 791

Q.E.D.

PROPOSITION XIV. THEOREM.

793. *In two polar triangles each angle of the one is the supplement of the opposite side in the other.*





Let ABC, A′B′C′ be two polar triangles. Let the letter at the vertex of each angle denote its value in degrees, and the small letter the value of the opposite side in degrees.

To prove that $A + a' = 180°$, $B + b' = 180°$, $C + c' = 180°$;
$A' + a = 180°$, $B' + b = 180°$, $C' + c = 180°$.

Proof. Produce the arcs AB, AC until they meet $B'C'$ at the points D, E, respectively.

Now B' is the pole of AE. $\therefore B'E = 90°$. § 761

Also C' is the pole of AD. $\therefore C'D = 90°$.

Adding, $B'E + C'D = 180°$. Ax. 2

That is, $B'D + DE + C'D = 180°$.

Or $DE + B'C' = 180°$.

But DE is the measure of the $\angle A$, § 779

and $B'C' = a'$.

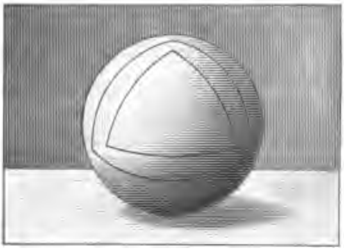
$\therefore A + a' = 180°$.

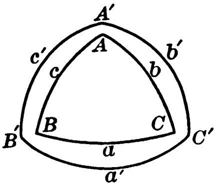
In a similar way all the other relations are proved. Q.E.D.

794. DEF. Polar triangles are often called **supplemental triangles.**

PROPOSITION XV. THEOREM.

795. *The sum of the angles of a spherical triangle is greater than 180° and less than 540°.*





Let ABC be a spherical triangle, and let A, B, C denote the values of its respective angles, and a′, b′, c′ the values of the opposite sides in the polar triangle A′B′C′.

To prove that $A + B + C > 180°$ *and* $< 540°$.

Proof. Since the $\triangle$ ABC, $A'B'C'$, are polar $\triangle$,

$A + a' = 180°$, $B + b' = 180°$, $C + c' = 180°$. § 793

$\therefore A + B + C + a' + b' + c' = 540°$. Ax. 2

$\therefore A + B + C = 540° - (a' + b' + c')$.

Now $a' + b' + c'$ is less than 360°. § 790

$\therefore A + B + C = 540°$ — some value less than 360°.

$\therefore A + B + C > 180°$.

Again $a' + b' + c'$ is greater than 0°.

$\therefore A + B + C < 540°$. Q. E. D.

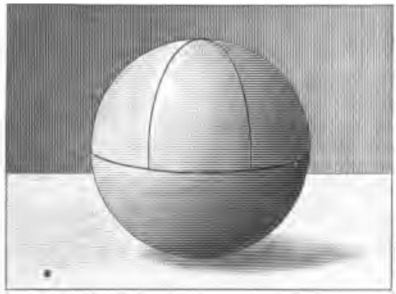
796. COR. *A spherical triangle may have two, or even three, right angles; and it may have two, or even three, obtuse angles.*

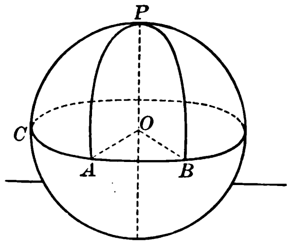
797. DEF. A spherical triangle having two right angles is called a **bi-rectangular** triangle; and a spherical triangle having three right angles is called a **tri-rectangular** triangle.

798. DEF. The **spherical excess** of a triangle is the difference between the sum of its angles and 180°.

PROPOSITION XVI. THEOREM.

799. *In a bi-rectangular spherical triangle the sides opposite the right angles are quadrants, and the side opposite the third angle measures that angle.*





Let PAB be a bi-rectangular triangle, with A, B right angles.

To prove that PA and PB are quadrants, and that the $\angle P$ is measured by the arc AB.

Proof. Since the $\angle$s A and B are right angles, the planes of the arcs PA, PB are $\perp$ to the plane of the arc AB. § 780

$\therefore PA$ and PB must each pass through the pole of AB. § 754

$\therefore P$ is the pole of AB, and PA, PB are quadrants. § 761

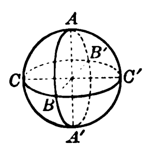
Also the $\angle P$ is measured by the arc AB. § 779

Q.E.D.

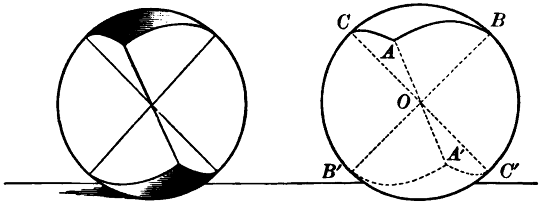
800. COR. 1. *If two sides of a spherical triangle are quadrants, the third side measures the opposite angle.*

801. COR. 2. *Each side of a tri-rectangular spherical triangle is a quadrant.*

802. COR. 3. *Three planes passed through the centre of a sphere, each perpendicular to the other two planes, divide the surface of the sphere into eight equal tri-rectangular triangles.*



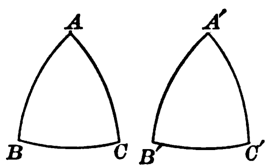
803. Def. If through the centre O of a sphere three diameters AA', BB', CC' are drawn, and the points A, B, C are joined by arcs of great circles, and also the points A', B', C', the two spherical triangles ABC and $A'B'C'$ are called **symmetrical** spherical triangles.



In the same way we may form two symmetrical polygons of any number of sides, and place each of them in any position we choose upon the surface of the sphere.

804. Two symmetrical triangles are mutually equilateral and equiangular; yet in general they cannot be made to coincide by superposition. If in the above figure the triangle ABC is made to slide on the surface of the sphere until the vertex A falls on A', it will be evident that the two triangles cannot be made to coincide and that the corresponding parts of the triangles occur in **reverse order.**

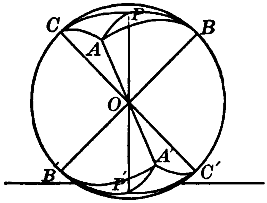
805. If, however, $AB = AC$, and $A'B' = A'C'$; that is, if the two symmetrical triangles are *isosceles*, then, because AB, AC, $A'B'$, $A'C'$ are all equal, and the angles A and A' are equal, being opposite dihedral angles (§ 803), the two triangles can be made to coincide. Therefore,



806. *If two symmetrical spherical triangles are isosceles, they are superposable and therefore equal.*

PROPOSITION XVII. THEOREM.

807. *Two symmetrical spherical triangles are equivalent.*



Let ABC, A′B′C′ be two symmetrical spherical triangles with their homologous vertices opposite each to each.

To prove that the triangles ABC, $A'B'C'$ are equivalent.

Proof. Let P be the pole of a small circle passing through the points A, B, C, and let POP' be a diameter.

Draw the great circle arcs PA, PB, PC, $P'A'$, $P'B'$, $P'C'$.

Then $PA = PB = PC$. § 758

Now $P'A' = PA,\ P'B' = PB,\ P'C' = PC$. § 804

$\therefore P'A' = P'B' = P'C'$. Ax. 1

$\therefore$ the two symmetrical $\triangle$s PAC and $P'A'C'$ are isosceles.

$\therefore \triangle PAC = \triangle P'A'C'$. § 806

Similarly, $\triangle PAB = \triangle P'A'B'$,

and $\triangle PBC = \triangle P'B'C'$.

Now $\triangle ABC \approx \triangle PAC + \triangle PAB + \triangle PBC$, Ax. 9

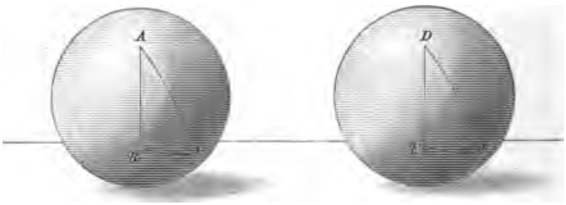
and $\triangle A'B'C' \approx \triangle P'A'C' + \triangle P'A'B' + \triangle P'B'C'$.

$\therefore \triangle ABC \approx \triangle A'B'C'$. Q. E. D.

If the pole P should fall without the $\triangle ABC$, then P' would fall without $\triangle A'B'C'$, and each triangle would be equivalent to the sum of two symmetrical isosceles triangles diminished by the third; so that the result would be the same as before.

Proposition XVIII. Theorem.

808. *Two triangles on the same sphere or equal spheres are equal, if two sides and the included angle, or two angles and the included side, of the one are respectively equal to the corresponding parts of the other and arranged in the same order.*

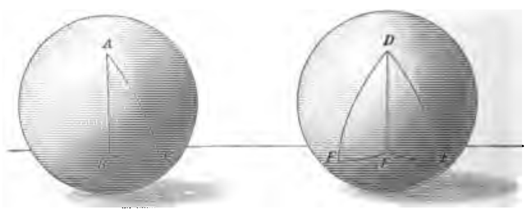


Proof. By superposition, as in plane ▵. §§ 143, 139

Q. E. D.

Proposition XIX. Theorem.

809. *Two triangles on the same sphere or equal spheres are symmetrical, if two sides and the included angle, or two angles and the included side, of the one are equal, respectively, to the corresponding parts of the other and arranged in the reverse order.*



Proof. Construct the △ DEF' symmetrical with respect to the △ DEF upon the same sphere.

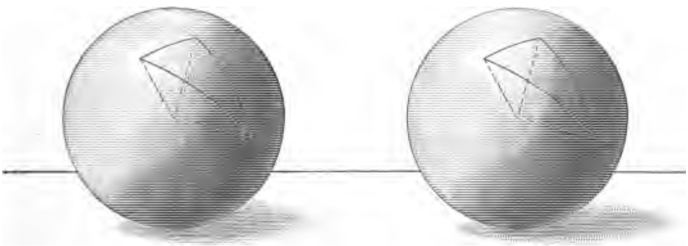
Then $\triangle ABC$ can be superposed upon the $\triangle DEF'$, so that they will coincide as in the corresponding case of plane $\triangle$s.

But $\triangle$s DEF' and DEF are symmetrical by construction.

$\therefore$ $\triangle ABC$, which coincides with $\triangle DEF'$, is symmetrical with respect to $\triangle DEF$. Q.E.D.

Proposition XX. Theorem.

810. *Two mutually equilateral triangles on the same sphere or equal spheres are mutually equiangular, and are equal or symmetrical.*



Proof. The face $\angle$s of the corresponding trihedral $\angle$s at the centre of the sphere are equal respectively. § 237

Therefore, the corresponding dihedral $\angle$s are equal. § 583

Hence, the $\angle$s of the spherical $\triangle$s are respectively equal.

Therefore, the $\triangle$s are equal or symmetrical, according as their equal sides are arranged in the same or reverse order. Q.E.D.

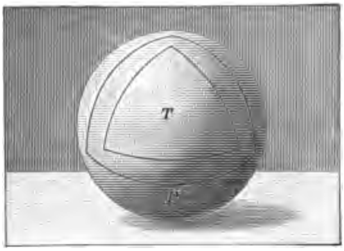
Ex. 737. The radius of a sphere is 4 inches. From any point on the surface as a pole a circle is described upon the sphere with an opening of the compasses equal to 3 inches. Find the area of this circle.

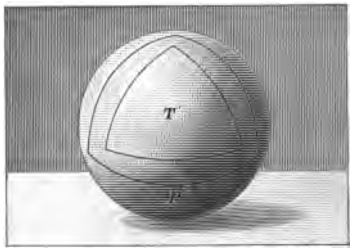
Ex. 738. The edge of a regular tetrahedron is a. Find the radii R, R' of the inscribed and circumscribed spheres.

Ex. 739. Find the diameter of the section of a sphere 10 inches in diameter made by a plane 3 inches from the centre.

Proposition XXI. Theorem.

811. *Two mutually equiangular triangles on the same sphere or equal spheres are mutually equilateral, and are either equal or symmetrical.*





Let the spherical triangles T and T′ be mutually equiangular.

To prove that T and T' are mutually equilateral, and equal or symmetrical.

Proof. Let the $\triangle P$ be the polar $\triangle$ of T, and P' of T'.

By hypothesis, the $\triangle$s T and T' are mutually equiangular.

$\therefore$ the polar $\triangle$s P and P' are mutually equilateral. § 793

$\therefore$ the polar $\triangle$s P and P' are mutually equiangular. § 810

But the $\triangle$s T and T' are the polar $\triangle$s of P and P'. § 792

$\therefore$ the $\triangle$s T and T' are mutually equilateral. § 793

Hence, the $\triangle$s T and T' are equal or symmetrical. § 810

Q. E. D.

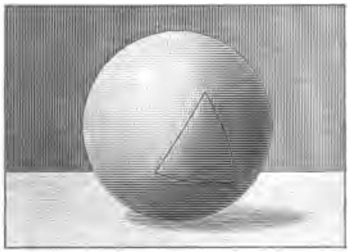
Note. The statement that mutually equiangular spherical triangles are mutually equilateral, and equal or symmetrical, is true only when they are on the same sphere, or equal spheres. But when the spheres are unequal, the spherical triangles are unequal; and the ratio of their homologous sides is equal to the ratio of the radii of the spheres. § 465

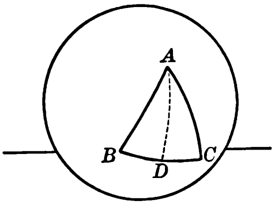
Ex. 740. At a given point in a given arc of a great circle, to construct a spherical angle equal to a given spherical angle.

Ex. 741. To inscribe a circle in a given spherical triangle.

PROPOSITION XXII. THEOREM.

812. *In an isosceles spherical triangle, the angles opposite the equal sides are equal.*





In the spherical triangle ABC, let AB equal AC.

To prove that $\angle B = \angle C$.

Proof. Draw the arc AD of a great circle, from the vertex A to the middle of the base BC.

Then $\triangle ABD$ and ACD are mutually equilateral.

$\therefore \triangle ABD$ and ACD are mutually equiangular. § 810

$\therefore \angle B = \angle C$. Q. E. D.

813. COR. *The arc of a great circle drawn from the vertex of an isosceles spherical triangle to the middle of the base bisects the vertical angle, is perpendicular to the base, and divides the triangle into two symmetrical triangles.*

Ex. 742. To circumscribe a circle about a given spherical triangle.

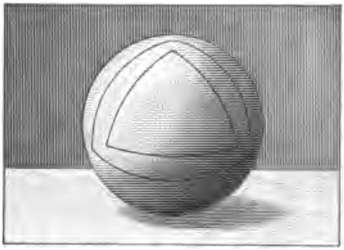
Ex. 743. Given a spherical triangle whose sides are 60°, 80°, and 100°. Find the angles of its polar triangle.

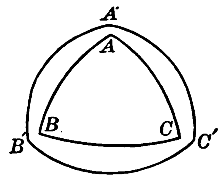
Ex. 744. Given a spherical triangle whose angles are 70°, 75°, and 95°. Find the sides of its polar triangle.

Ex. 745. Find the ratio of two homologous sides of two mutually equiangular triangles on spheres whose radii are 12 inches and 20 inches.

Proposition XXIII. Theorem.

814. *If two angles of a spherical triangle are equal, the sides opposite these angles are equal and the triangle is isosceles.*





In the spherical triangle ABC, let the angle B equal the angle C.

To prove that $AC = AB$.

Proof. Let the $\triangle A'B'C'$ be the polar $\triangle$ of the $\triangle ABC$.

Now $\angle B = \angle C$. Hyp.

$\therefore A'C' = A'B'$. § 793

$\therefore \angle B' = \angle C'$. § 812

$\therefore AC = AB$. § 793

Q. E. D.

Ex. 746. To bisect a spherical angle.

Ex. 747. To construct a spherical triangle, having given two sides and the included angle.

Ex. 748. To construct a spherical triangle, having given two angles and the included side.

Ex. 749. To construct a spherical triangle, having given the three sides.

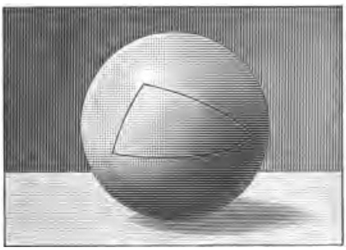
Ex. 750. To construct a spherical triangle, having given the three angles.

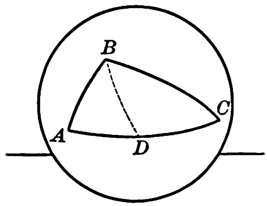
Ex. 751. To pass a plane tangent to a sphere at a given point on the surface of the sphere.

Ex. 752. To pass a plane tangent to a sphere through a given straight line without the sphere.

Proposition XXIV. Theorem.

815. *If two angles of a spherical triangle are unequal, the sides opposite are unequal, and the greater side is opposite the greater angle;* CONVERSELY, *if two sides are unequal, the angles opposite are unequal, and the greater angle is opposite the greater side.*





1. **In the triangle ABC, let the angle ABC be greater than the angle ACB.**

To prove that $AC > AB$.

Proof. Draw the arc BD of a great circle, making $\angle CBD$ equal $\angle ACB$. Then $DC = DB$. § 814

Now $AD + DB > AB$. § 789

$$\therefore AD + DC > AB, \text{ or } AC > AB.$$

2. **Let AC be greater than AB.**

To prove that the $\angle ABC$ *is greater than the* $\angle ACB$.

Proof. The $\angle ABC$ must be equal to, less than, or greater than the $\angle ACB$.

If $\angle ABC = \angle C$, then $AC = AB$; § 814

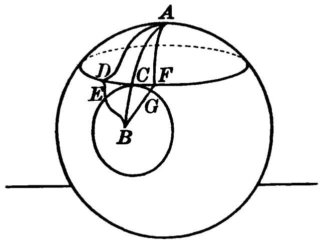
and if $\angle ABC$ is less than $\angle C$, then $AC < AB$. (1)

But both of these conclusions are contrary to the hypothesis.

$\therefore \angle ABC$ is greater than $\angle C$. Q. E. D.

Proposition XXV. Theorem.

816. *The shortest line that can be drawn on the surface of a sphere between two points is the arc of a great circle, joining the two points not greater than a semicircumference.*



Let AB be the arc of a great circle, not greater than a semicircumference, joining the points A and B.

To prove that AB is the shortest line that can be drawn on the surface joining A and B.

Proof. Let C be any point in AB.

With A and B as poles and AC and BC as polar distances, describe two arcs DCF and ECG.

The arcs DCF and ECG have only the point C in common. For if F is any other point in DCF, and if arcs of great circles AF and BF are drawn, then

$$AF = AC. \qquad \S\ 758$$

But $$AF + BF > AC + BC. \qquad \S\ 789$$

Take away AF from the left member of the inequality, and its equal AC from the right member.

Then $$BF > BC. \qquad \text{Ax. 5}$$

Therefore, $$BF > BG, \text{ the equal of } BC.$$

Hence, F lies without the circumference whose pole is B, and the arcs DCF and ECG have only the point C in common.

Now let $ADEB$ be any line from A to B on the surface of the sphere, which does not pass through C.

This line will cut the arcs DCF and ECG in separate points D and E, and if we revolve the line AD about A as a fixed point until D coincides with C we shall have a line from A to C equal to the line AD.

In like manner, we can draw a line from B to C equal to the line BE.

Therefore, a line can be drawn from A to B through C that is equal to the sum of the lines AD and BE, and hence less than the line $ADEB$ by the line DE.

Therefore, no line which does not pass through C can be the shortest line from A to B.

Therefore, the shortest line from A to B passes through C.

But C is *any* point in the arc AB.

Therefore, the shortest line from A to B passes through *every* point of the arc AB, and consequently coincides with the arc AB.

Therefore, the shortest line from A to B is the great circle arc AB. Q.E.D.

Ex. 753. The three medians of a spherical triangle meet in a point.

Ex. 754. To construct a spherical surface with a given radius that passes through three given points.

Ex. 755. To construct a spherical surface with a given radius that passes through two given points and is tangent to a given plane.

Ex. 756. To construct a spherical surface with a given radius that passes through two given points and is tangent to a given sphere.

Ex. 757. All arcs of great circles drawn through a pole of a given great circle are perpendicular to the circumference of the great circle.

Ex. 758. The smallest circle whose plane passes through a given point within a sphere is the one whose plane is perpendicular to the radius through the given point.

MEASUREMENT OF SPHERICAL SURFACES.

817. Def. A **zone** is a portion of the surface of a sphere included between two parallel planes.

The circumferences of the sections made by the planes are called the **bases of the zone**, and the distance between the planes is the **altitude of the zone**.

818. Def. A **zone of one base** is a zone one of whose bounding planes is tangent to the sphere.

If a great circle $PADQ$ (Fig. 1) is revolved about its diameter PQ, the arc AD will generate a zone, the points A and D will generate its bases, and CF will be its altitude.

The arc PA will generate a zone of one base.

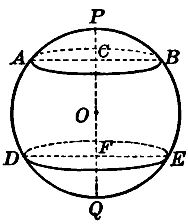


Fig. 1.

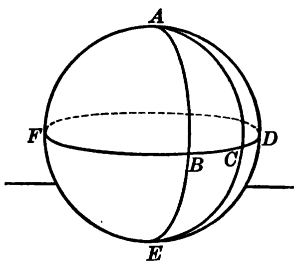


Fig. 2.

819. Def. A **lune** is a portion of the surface of a sphere bounded by two semicircumferences of great circles.

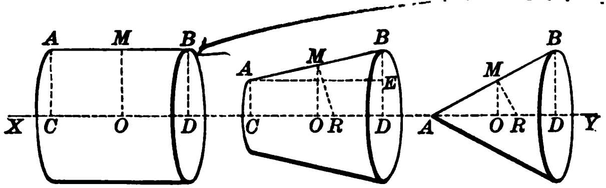
820. Def. The **angle** of a lune is the angle between the semicircumferences which form its boundaries.

Thus (Fig. 2), $ABECA$ is a lune, BAC is its angle.

821. Def. It is convenient to divide each of the eight equal tri-rectangular triangles of which the surface of a sphere is composed (§ 802) into 90 equal parts, and to call each of these parts a **spherical degree**. The surface of every sphere, therefore, contains 720 spherical degrees.

PROPOSITION XXVI. THEOREM.

822. *The area of the surface generated by a straight line revolving about an axis in its plane is equal to the product of the projection of the line on the axis by the circumference whose radius is a perpendicular erected at the middle point of the line and terminated by the axis.* Smooth



Let XY be the axis, AB the revolving line, M its middle point, CD its projection on XY, MO perpendicular to XY, and MR to AB.

To prove that the area $AB = CD \times 2\pi MR$.

Proof. 1. If AB is ∥ to XY, $CD = AB$, MR coincides with MO, and the area AB is the surface of a right cylinder. § 697

2. If AB is not ∥ to XY, and does not cut XY, the area AB is the surface of the frustum of a cone of revolution.

$\therefore$ the area $AB = AB \times 2\pi MO$. § 728

Draw AE ∥ to XY.

The $\triangle$ ABE and MOR are similar. § 359

$\therefore MO : AE = MR : AB$. § 351

$\therefore AB \times MO = AE \times MR$. § 327

Or $AB \times MO = CD \times MR$.

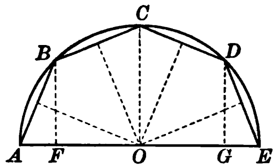
Substituting, the area $AB = CD \times 2\pi MR$.

3. If A lies in the axis XY, the reasoning still holds, but AE and CD coincide, and the truth follows from § 722.

Q. E. D.

PROPOSITION XXVII. THEOREM.

823. *The area of the surface of a sphere is equal to the product of its diameter by the circumference of a great circle.*



Let S denote the surface, R the radius, of a sphere generated by the semicircle ABCDE revolving about the diameter AE as an axis.

To prove that $S = AE \times 2\pi R.$

Proof. Inscribe in the semicircle half of a regular polygon having an *even* number of sides, as $ABCDE$.

From the centre draw ⊥s to the chords AB, BC, etc.

These ⊥s bisect the chords (§ 245) and are equal. § 249

Let a denote the length of each of these ⊥s.

From B, C, and D drop the ⊥s BF, CO, and DG to AE.

Then the area $AB = AF \times 2\pi a,$ § 822

the area $BC = FO \times 2\pi a$, etc.

$\therefore$ the area $ABCDE = AE \times 2\pi a.$

Denote the area of the surface described by the semi-polygon by S', and let the number of sides of the semi-polygon be indefinitely increased.

Then S' approaches S as a limit,

and a approaches R as a limit. § 449

$\therefore AE \times 2\pi a$ approaches $AE \times 2\pi R$ as a limit. § 279

But $S' = AE \times 2\pi a$, always. § 822

$\therefore S = AE \times 2\pi R.$ § 284

Q.E.D.

824. COR. 1. *The surface of a sphere is equivalent to four great circles; that is, to* $4\pi R^2$.

For πR^2 is equal to the area of a great circle, § 463
and $4\pi R^2$ is equal to $2R \times 2\pi R$, the area of the surface of a sphere. § 823

825. COR. 2. *The areas of the surfaces of two spheres are as the squares of their radii, or as the squares of their diameters.*

Let R and R' denote the radii, D and D' the diameters, and S and S' the areas of the surfaces of two spheres.

Then $$\frac{S}{S'} = \frac{4\pi R^2}{4\pi R'^2} = \frac{R^2}{R'^2} = \frac{(\frac{1}{2}D)^2}{(\frac{1}{2}D')^2} = \frac{D^2}{D'^2}.$$

826. COR. 3. *The area of a zone is equal to the product of its altitude by the circumference of a great circle.*

If we apply the reasoning of § 823 to the zone generated by the revolution of the arc BCD, we obtain

$$\text{the area of zone } BCD = FG \times 2\pi R,$$

where FG is the altitude of the zone and $2\pi R$ the circumference of a great circle.

827. COR. 4. *Zones on the same sphere or equal spheres are to each other as their altitudes.*

828. COR. 5. *A zone of one base is equivalent to a circle whose radius is the chord of the generating arc.*

The arc AB generates a zone of one base;

and zone $AB = AF \times 2\pi R = \pi AF \times AE$.

But $AF \times AE = \overline{AB}^2$. § 370

$\therefore$ the zone $AB = \pi\overline{AB}^2$.

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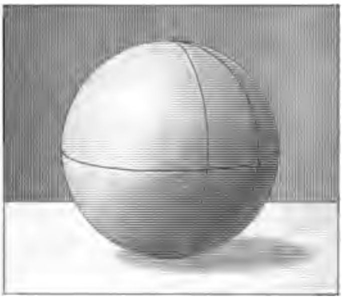
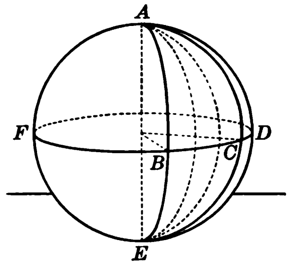
PROPOSITION XXVIII. THEOREM.

§ 829. *The area of a lune is to the area of the surface of the sphere as the number of degrees in the angle of the lune is to 360.*





Let ABEC be a lune, BCDF the great circle whose pole is A; also let A denote the number of degrees in the angle of the lune, L the area of the lune, and S the area of the surface of the sphere.

To prove that $L:S = A:360$.

Proof. The arc BC measures the $\angle A$ of the lune. § 779

Hence, arc BC : circumference $BCDF = A:360$.

If BC and $BCDF$ are commensurable, let their common measure be contained m times in BC, and n times in $BCDF$.

Then arc BC : circumference $BCDF = m:n$.

$\therefore A:360 = m:n$. § 288

Pass arcs of great ⓢ through the diameter AE and all the points of the division of $BCDF$. These arcs will divide the entire surface into n equal lunes, of which the lune $ABEC$ will contain m.

$\therefore L:S = m:n$.

$\therefore L:S = A:360$. Ax. 1

If BC and $BCDF$ are incommensurable, the theorem can be proved by the method of limits as in § 549. Q.E.D.

830. Cor. 1. *The number of spherical degrees in a lune is equal to twice the number of angle degrees in the angle of the lune.*

If L and S are expressed in spherical degrees, § 821

then $$L : 720 = A : 360.$$ § 829

Therefore, $$L = 2A.$$

831. Cor. 2. *The area of a lune is equal to one ninetieth of the area of a great circle multiplied by the number of degrees in the angle of the lune.*

For $$L : 4\pi R^2 = A : 360.$$ § 829

Therefore, $$L = \frac{\pi R^2 A}{90}.$$

832. Cor. 3. *Two lunes on the same sphere or equal spheres have the same ratio as their angles.*

For $$L : L' = \frac{\pi R^2 A}{90} : \frac{\pi R^2 A'}{90}.$$ § 831

That is, $$L : L' = A : A'.$$

833. Cor. 4. *Two lunes which have the equal angles, but are situated on unequal spheres, have the same ratio as the squares of the radii of the spheres on which they are situated.*

For $$L : L' = \frac{\pi R^2 A}{90} : \frac{\pi R'^2 A}{90}.$$ § 831

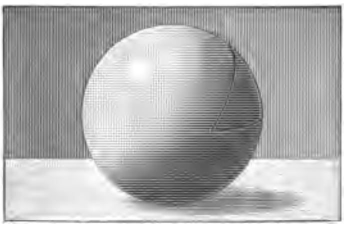
That is, $$L : L' = R^2 : R'^2.$$

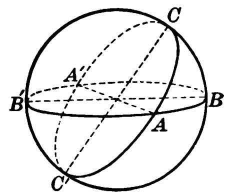
Ex. 759. Given the radius of a sphere 10 inches. Find the area of a lune whose angle is 30°.

Ex. 760. Given the diameter of a sphere 16 inches. Find the area of a lune whose angle is 75°.

PROPOSITION XXIX. THEOREM.

834. *The area of a spherical triangle, expressed in spherical degrees, is numerically equal to the spherical excess of the triangle.*





Let A, B, C denote the values of the angles of the spherical triangle ABC, and E the spherical excess.

To prove that the number of spherical degrees in $\triangle ABC = E$.

Proof. Produce the sides of the $\triangle ABC$ to complete circles.

These circles divide the surface of the sphere into eight spherical triangles, of which any four having a common vertex, as A, form the surface of a hemisphere.

The $\triangle$s $A'BC$, $AB'C'$ are symmetrical and equivalent. § 807

And $\quad \triangle ABC + \triangle A'BC \approx$ lune $ABA'C$.

Put the $\triangle AB'C'$ for its equivalent, the $\triangle A'BC$.

Then $\quad \triangle ABC + \triangle AB'C' \approx$ lune $ABA'C$.

Also $\quad \triangle ABC + \triangle AB'C \approx$ lune $BAB'C$.

And $\quad \triangle ABC + \triangle ABC' \approx$ lune $CAC'B$.

Add and observe that in spherical degrees

$$\triangle ABC + \triangle AB'C' + \triangle AB'C + \triangle ABC' = 360, \quad \text{§ 821}$$

and $\quad ABA'C + BAB'C + CAC'B = 2(A + B + C)$. § 830

Then $\quad 2\triangle ABC + 360 = 2(A + B + C)$.

$$\therefore \triangle ABC = A + B + C - 180 = E.$$

Q.E.D.

835. COR. 1. ***The area of a spherical triangle is to the area of the surface of the sphere as the number which expresses its spherical excess is to*** 720.

For the number of spherical degrees in a spherical $\triangle ABC$ is equal to E (§ 834), and the number of spherical degrees in S, the surface of the sphere, is equal to 720. § 821

$$\therefore \triangle ABC : S = E : 720.$$

836. COR. 2. ***The area of a spherical triangle is equal to the area of a great circle multiplied by the number of degrees in E divided by one hundred eighty.***

For $\triangle ABC : S = E : 720.$ § 835

But $S = 4\pi R^2.$ § 824

$$\therefore \triangle ABC = \frac{4\pi R^2 E}{720} = \frac{\pi R^2 E}{180}.$$

Ex. 761. What part of the surface of a sphere is a triangle whose angles are 120°, 100°, and 95°? What is its area in square inches, if the radius of the sphere is 6 inches?

Ex. 762. Find the area of a spherical triangle whose angles are 100°, 120°, 140°, if the diameter of the sphere is 16 inches.

Ex. 763. If the radii of two spheres are 6 inches and 4 inches respectively, and the distance between their centres is 5 inches, what is the area of the circle of intersection of these spheres?

Ex. 764. Find the radius of the circle determined in a sphere of 5 inches diameter by a plane 1 inch from the centre.

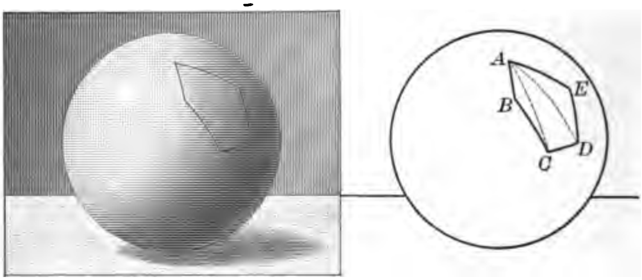
Ex. 765. If the radii of two concentric spheres are R and R', and if a plane is drawn tangent to the interior sphere, what is the area of the section made in the other sphere?

Ex. 766. Two points A and B are 8 inches apart. Find the locus in space of a point 5 inches from A and 7 inches from B.

Ex. 767. The radii of two parallel sections of the same sphere are a and b respectively, and the distance between these sections is d. Find the radius of the sphere.

Proposition XXX. Theorem.

837. *If T denotes the number which expresses the sum of the angles of a spherical polygon of n sides, the area of the polygon expressed in spherical degrees is numerically equal to $T - (n - 2)\,180$.*



Let ABCDE be a polygon of n sides.

To prove that the area of ABCDE expressed in spherical degrees is numerically equal to

$$T - (n - 2)\,180.$$

Proof. Divide the polygon into spherical triangles by drawing diagonals from any vertex, as A.

These diagonals divide the polygon into $n - 2$ spherical $\triangle$s. The area of each triangle in spherical degrees is numerically equal to the sum of its angles minus 180. § 834

Hence, the sum of the areas of all the $n - 2$ triangles expressed in spherical degrees is numerically equal to the sum of all their angles minus $(n - 2)\,180$.

Now the sum of the areas of the triangles is the area of the polygon, and the sum of the angles of the triangles is the sum of the angles of the polygon.

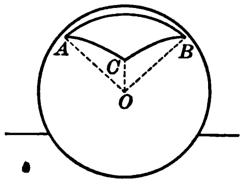
Therefore, the area of the polygon expressed in spherical degrees is numerically equal to $T - (n - 2)\,180$. Q. E. D.

SPHERICAL VOLUMES.

838. Def. A **spherical pyramid** is the portion of a sphere bounded by a spherical polygon and the planes of its sides.

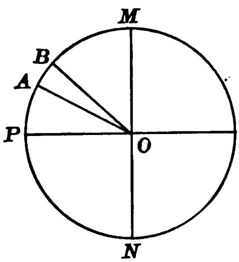
The centre of the sphere is the **vertex** of the pyramid, and the spherical polygon is the **base** of the pyramid.

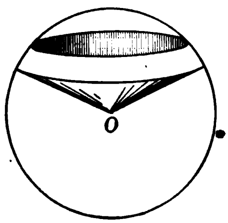
Thus, $O\text{-}ABC$ is a spherical pyramid.



839. Def. A **spherical sector** is the portion of a sphere generated by the revolution of a circular sector about any diameter of the circle of which the sector is a part.

The **base** of a spherical sector is the zone generated by the arc of the circular sector.





840. Def. A **spherical segment** is a portion of a sphere contained between two parallel planes.

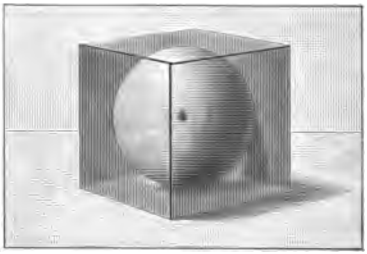
841. Def. The **bases** of a spherical segment are the sections made by the parallel planes, and the **altitude** of a spherical segment is the perpendicular distance between the bases.

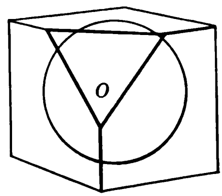
842. Def. If one of the parallel planes is tangent to the sphere, the segment is called a **segment of one base.**

843. Def. A **spherical wedge** is a portion of a sphere bounded by a lune and two great semicircles.

PROPOSITION XXXI. THEOREM.

844. *The volume of a sphere is equal to the product of the area of its surface by one third of its radius.*





Let R be the radius of a sphere whose centre is O, S its surface, and V its volume.

To prove that $V = S \times \frac{1}{3} R.$

Proof. Conceive a cube to be circumscribed about the sphere. Its volume is greater than that of the sphere, because it contains the sphere.

From O, the centre of the sphere, conceive lines to be drawn to the vertices of the cube.

These lines are the edges of six quadrangular pyramids, whose bases are the faces of the cube, and whose common altitude is the radius of the sphere.

The volume of each pyramid is equal to the product of its base by $\frac{1}{3}$ its altitude. Hence, the volume of the six pyramids, that is, the volume of the circumscribed cube, is equal to the area of the surface of the cube multiplied by $\frac{1}{3} R$.

Now conceive planes drawn tangent to the sphere, at the points where the edges of the pyramids cut its surface. We then have a circumscribed solid whose volume is nearer that of the sphere than is the volume of the circumscribed cube, because each tangent plane cuts away a portion of the cube.

From O conceive lines to be drawn to each of the polyhedral angles of the solid thus formed. These lines form the edges of a series of pyramids, whose bases are together equal to the surface of the solid, and whose common altitude is the radius of the sphere; and the volume of each pyramid thus formed is equal to the product of its base by $\frac{1}{3}$ its altitude.

Hence, the sum of the volumes of these pyramids, that is, the volume of this new solid, is again equal to the area of its surface multiplied by $\frac{1}{3}R$.

If we denote the area of the surface of this polyhedron by S', and the volume of the polyhedron by V',

$$V' = S' \times \tfrac{1}{3}R.$$

If we continue to draw tangent planes to the sphere, we continue to diminish the circumscribed solid, since each new plane cuts off a corner of the polyhedron.

By continuing this process indefinitely, we can make the difference between the volumes of the circumscribed solid and sphere less than any assigned quantity, however small, but we cannot make it zero; and the difference between the areas of the surfaces of the circumscribed solid and sphere less than any assigned quantity, however small, but we cannot make it zero.

Hence, V is the limit of V', and S is the limit of S'. § 275

But $V' = S' \times \frac{1}{3}R$, always.

$\therefore V = S \times \frac{1}{3}R.$ § 284

Q. E. D.

845. Cor. 1. *The volume of a sphere is equal to* $4\pi R^2 \times \frac{1}{3}R$; *that is,* $\frac{4}{3}\pi R^3$, *or* $\frac{1}{6}\pi D^3$. ($D =$ diameter.)

846. Cor. 2. *The volumes of two spheres are to each other as the cubes of their radii.*

For $V : V' = \frac{4}{3}\pi R^3 : \frac{4}{3}\pi R'^3 = R^3 : R'^3$.

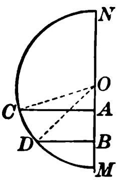
847. Cor. 3. *The volume of a spherical pyramid is equal to the product of its base by one third of the radius of the sphere.*

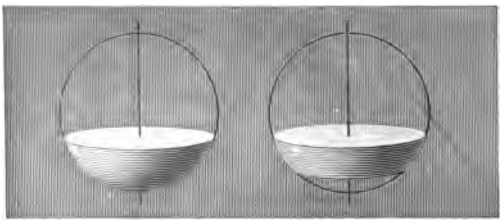
848. Cor. 4. *The volume of a spherical sector is equal to one third the product of the zone which forms its base by the radius of the sphere.*

If R denotes the radius of the sphere, C the circumference of a great circle, H the altitude of the zone, Z the surface of the zone, and V the volume of the sector; then, $C = 2\pi R$ (§ 458), $Z = 2\pi R \times H$ (§ 826), and $V = 2\pi RH \times \frac{1}{3}R = \frac{2}{3}\pi R^2 H$.

Proposition XXXII. Problem.

849. *To find the volume of a spherical segment.*





Let AC and BD be two semi-chords perpendicular to the diameter MN of the semicircle NCDM. Let OM be equal to R, AM to a, BM to b, AB to h, AC to r, BD to r′.

Case 1. *To find the volume of the segment of one base generated by the circular semi-segment ACM, as the semicircle revolves about MN as an axis.*

The sector generated by $OCM = \frac{2}{3}\pi R^2 a$. § 848

The cone generated by $OCA = \frac{1}{3}\pi r^2 (R - a)$. § 724

Hence, segment $ACM = \frac{2}{3}\pi R^2 a - \frac{1}{3}\pi r^2 (R - a)$

$$= \frac{\pi}{3}(2R^2 a - Rr^2 + ar^2).$$

Now $r^2 = AM \times AN = a(2R - a)$. § 370

$$\therefore \text{ the segment } ACM = \pi a^2\left(R - \frac{a}{3}\right). \qquad (1)$$

If from the relation $r^2 = a(2R - a)$ we find the value of R, and substitute it in (1), we obtain the volume in terms of the altitude and the radius of the base.

$$\text{The segment } ACM = \tfrac{1}{2}\pi r^2 a + \tfrac{1}{6}\pi a^3. \qquad (2)$$

CASE 2. *To find the volume of the segment of two bases generated by the circular semi-segment ABDC, as the semicircle revolves about NM as an axis.*

Since the volume is obviously the difference of the volumes of the segments of one base generated by the circular semi-segments ACM and BDM, therefore, by formula (1),

$$\text{segment } ABDC = \pi a^2\left(R - \frac{a}{3}\right) - \pi b^2\left(R - \frac{b}{3}\right)$$

$$= \pi R(a^2 - b^2) - \frac{\pi}{3}(a^3 - b^3) \qquad (3)$$

(put h for $a - b$) $$= \pi Rh(a + b) - \frac{\pi h}{3}(a^2 + ab + b^2)$$

$$= \pi h\left[(Ra + Rb) - \tfrac{1}{3}(a^2 + ab + b^2)\right].$$

Now $a - b = h.\quad \therefore a^2 - 2ab + b^2 = h^2.$

Add $3ab$ to each side, $a^2 + ab + b^2 = h^2 + 3ab.$

Since $(2R - a)a = r^2$, and $(2R - b)b = r'^2$, § 370

$$Ra + Rb = \frac{r^2 + r'^2}{2} + \frac{a^2 + b^2}{2}.$$

$$\therefore \text{the segment } ABDC = \pi h\left[\frac{r^2 + r'^2}{2} + \frac{a^2 + b^2}{2} - \frac{h^2}{3} - ab\right]$$

$$= \pi h\left[\frac{r^2 + r'^2}{2} + \frac{h^2}{2} + ab - \frac{h^2}{3} - ab\right]$$

$$= \frac{h}{2}(\pi r^2 + \pi r'^2) + \frac{\pi h^3}{6}.$$

Q.E.F.

NUMERICAL EXERCISES.

Ex. 768. Find the surface of a sphere, if the diameter is (i) 10 inches; (ii) 1 foot 9 inches; (iii) 2 feet 4 inches; (iv) 7 feet; (v) 10.5 feet.

Ex. 769. Find the diameter of a sphere if the surface is (i) 616 square inches; (ii) $38\frac{1}{2}$ square feet; (iii) 9856 square feet.

Ex. 770. The circumference of a dome in the shape of a hemisphere is 66 feet. How many square feet of lead are required to cover it?

Ex. 771. If the ball on the top of St. Paul's Cathedral in London is 6 feet in diameter, what would it cost to gild it at 7 cents per square inch?

Ex. 772. What is the numerical value of the radius of a sphere, if its surface has the same numerical value as the circumference of a great circle?

Ex. 773. Find the surface of a lune, if its angle is 30°, and the total surface of the sphere is 4 square feet.

Ex. 774. What fractional part of the whole surface of a sphere is a spherical triangle whose angles are 43° 27′, 81° 57′, and 114° 36′?

Ex. 775. The angles of a spherical triangle are 60°, 70°, and 80°. The radius of the sphere is 14 feet. Find the area of the triangle.

Ex. 776. The sides of a spherical triangle are 80°, 74°, and 128°. The radius of the sphere is 14 feet. Find the area of the polar triangle in square feet.

Ex. 777. Find the area of a spherical polygon on a sphere whose radius is $10\frac{1}{2}$ feet, if its angles are 100°, 120°, 140°, and 160°.

Ex. 778. The planes of the faces of a quadrangular spherical pyramid make with each other angles of 80°, 100°, 120°, and 150°; and the length of a lateral edge of the pyramid is 42 feet. Find the area of its base in square feet.

Ex. 779. The planes of the faces of a triangular spherical pyramid make with each other angles of 60°, 80°, and 100°, and the area of the base of the pyramid is 4π square feet. Find the radius of the sphere.

Ex. 780. The diameter of a sphere is 21 feet. Find the curved surface of a segment whose height is 5 feet.

Ex. 781. In a sphere whose radius is R, find the height of a zone whose area is equal to that of a great circle.

Ex. 782. What is the area of a zone of one base whose height is h, and the radius of the base r? What would be the area if the height were twice as great?

Ex. 783. The altitude of the torrid zone is 3200 miles. Find its area, assuming the earth to be a sphere with a radius of 4000 miles.

Ex. 784. A plane divides the surface of a sphere of radius R into two zones, such that the surface of the greater is the mean proportional between the entire surface and the surface of the smaller. Find the distance of the plane from the centre of the sphere.

Ex. 785. If a sphere of radius R is cut by two parallel planes equally distant from the centre, so that the area of the zone comprised between the planes is equal to the sum of the areas of its bases, find the distance of either plane from the centre.

Ex. 786. Find the area of the zone generated by an arc of $30°$, of which the radius is r, and which turns around a diameter passing through one of its extremities.

Ex. 787. Find the area of the zone of a sphere of radius R, illuminated by a lamp placed at the distance h from the sphere.

Ex. 788. How much of the earth's surface would a man see if he were raised to the height of the radius above it?

Ex. 789. To what height must a man be raised above the earth in order that he may see one sixth of its surface?

Ex. 790. The square on the diameter of a sphere and the square on an edge of the inscribed cube are as $3:1$.

Ex. 791. Find the volume of a sphere, if the diameter is (i) 13 inches; (ii) 3 feet 6 inches; (iii) 10 feet 6 inches; (iv) 14.7 feet.

Ex. 792. Find the diameter of a sphere, if the volume is (i) 75 cubic feet 1377 cubic inches; (ii) 179 cubic feet 1152 cubic inches; (iii) 1047.816 cubic feet; (iv) 38.808 cubic yards.

Ex. 793. Find the volume of a sphere whose circumference is 45 feet.

Ex. 794. Find the volume V of a sphere in terms of the circumference C of a great circle.

Ex. 795. Find the radius R of a sphere, having given the volume V.

Ex. 796. Find the radius R of a sphere, if its circumference and its volume have the same numerical value.

Ex. 797. The volume of a sphere is to the volume of the circumscribed cube as π is to 6.

Ex. 798. An iron ball 4 inches in diameter weighs 9 pounds. Find the weight of an iron shell 2 inches thick, whose external diameter is 20 inches.

Ex. 799. The radius of a sphere is 7 feet. What is the volume of a wedge whose angle is 36°?

Ex. 800. What is the angle of a spherical wedge, if its volume is one cubic foot, and the volume of the entire sphere is 6 cubic feet?

Ex. 801. Find the volume of a spherical sector, if the area of the zone of its base is 3 square feet, and the radius of the sphere is 1 foot.

Ex. 802. The radius of the base of a segment of a sphere is 16 inches, and the radius of the sphere is 20 inches. Find the volume of the segment.

Ex. 803. The inside of a wash-basin is in the shape of the segment of a sphere; the distance across the top is 16 inches, and its greatest depth is 6 inches. Find how many pints of water it will hold, reckoning $7\frac{1}{2}$ gallons to the cubic foot.

Ex. 804. What is the height of a zone, if its area is S, and the volume of the sphere to which it belongs is V?

Ex. 805. The radii of the bases of a spherical segment are 6 feet and 8 feet, and its height is 3 feet. Find its volume.

Ex. 806. Find the volume of a triangular spherical pyramid, if the angles of the spherical triangle which forms its base are each 100°, and the radius of the sphere is 7 feet.

Ex. 807. The circumference of a sphere is 28π feet. Find the volume of that part of the sphere included by the faces of a trihedral angle at the centre, the dihedral angles of which are 80°, 105°, and 140°.

Ex. 808. The planes of the faces of a quadrangular spherical pyramid make with each other angles of 80°, 100°, 120°, and 150°, and a lateral edge of the pyramid is $3\frac{1}{2}$ feet. Find the volume of the pyramid.

Ex. 809. Having given the volume V, and the height h, of a spherical segment of one base, find the radius R of the sphere.

Ex. 810. Find the weight of a sphere of radius R, which floats in a liquid of specific gravity s, with one fourth of its surface above the surface of the liquid. (The weight of a floating body is equal to the weight of the liquid displaced.)

MISCELLANEOUS EXERCISES.

Ex. 811. Determine a point in a given plane such that the difference of its distances from two given points on opposite sides of the plane shall be a maximum.

Ex. 812. The portion of a tetrahedron cut off by a plane parallel to any face is a tetrahedron similar to the given tetrahedron.

Ex. 813. Two symmetrical tetrahedrons are equivalent.

Ex. 814. Two symmetrical polyhedrons may be decomposed into the same number of tetrahedrons symmetrical each to each.

Ex. 815. Two symmetrical polyhedrons are equivalent.

Ex. 816. If a solid has two planes of symmetry perpendicular to each other, the intersection of these planes is an axis of symmetry of the solid.

Ex. 817. If a solid has three planes of symmetry perpendicular to one another, the three intersections of these planes are three axes of symmetry of the solid; and the common intersection of these axes is the centre of symmetry of the solid.

Ex. 818. The volume of a sphere is to the volume of the inscribed cube as π is to $\frac{2}{3}\sqrt{3}$.

Ex. 819. Find the area of the surface of the sphere inscribed in a regular tetrahedron whose edge is 6 inches.

Ex. 820. If a zone of one base is the mean proportional between the remainder of the surface of the sphere and the total surface of the sphere, find the distance of the base of the zone from the centre of the sphere.

Ex. 821. Find the difference between the volume of a frustum of a pyramid and the volume of a prism each 24 feet high, if the bases of the frustum are squares with sides 20 feet and 16 feet, respectively, and the base of the prism is the section of the frustum parallel to the bases and midway between them.

Ex. 822. If the earth is assumed to be a sphere of 4000 miles radius, how far at sea can a lighthouse 100 feet high be seen?

Ex. 823. If the atmosphere extends 50 miles above the surface of the earth, and the earth is assumed to be a sphere of 4000 miles radius, find the volume of the atmosphere.

Ex. 824. Draw a line through the vertex of any trihedral angle, making equal angles with its edges.

Ex. 825. In any trihedral angle, the three planes passed through the edges and the respective bisectors of the opposite face angles intersect in the same straight line.

Ex. 826. In any trihedral angle, the three planes passed through the bisectors of the face angles, perpendicular to these faces, respectively, intersect in the same straight line.

Ex. 827. In any trihedral angle, the three planes passed through the edges, perpendicular to the opposite faces, respectively, intersect in the same straight line.

Ex. 828. In a tetrahedron, the planes passed through the three lateral edges and the middle points of the opposite sides of the base intersect in a straight line.

Ex. 829. The lines drawn from each vertex of a tetrahedron to the point of intersection of the medians of the opposite face all meet in a point called the *centre of gravity*, which divides each line so that the shorter segment is to the whole line in the ratio 1 : 4.

Ex. 830. The straight lines joining the middle points of the opposite edges of a tetrahedron all pass through the centre of gravity of the tetrahedron, and are bisected by the centre of gravity.

Ex. 831. The plane which bisects a dihedral angle of a tetrahedron divides the opposite edges into segments proportional to the areas of the faces that include the dihedral angle.

Ex. 832. The altitude of a regular tetrahedron is equal to the sum of the four perpendiculars let fall from any point within the tetrahedron upon the four faces.

Ex. 833. Within a given tetrahedron, to find a point such that the planes passed through this point and the edges of the tetrahedron shall divide the tetrahedron into four equivalent tetrahedrons.

Ex. 834. To cut a cube by a plane so that the section shall be a regular hexagon.

Ex. 835. Two tetrahedrons are similar if a dihedral angle of one is equal to a dihedral angle of the other, and the faces that include these angles are respectively similar, and similarly placed.

Ex. 836. To cut a tetrahedral angle so that the section shall be a parallelogram.

Ex. 837. Two polyhedrons composed of the same number of tetrahedrons, similar each to each and similarly placed, are similar.

Ex. 838. If the homologous faces of two similar pyramids are respectively parallel, the straight lines which join the homologous vertices of the pyramids meet in a point.

Ex. 839. The volume of a right circular cylinder is equal to the product of the lateral area by half the radius.

Ex. 840. The volume of a right circular cylinder is equal to the product of the area of the rectangle which generates it, by the length of the circumference generated by the point of intersection of the diagonals of the rectangle.

Ex. 841. If the altitude of a right circular cylinder is equal to the diameter of the base, the volume is equal to the total area multiplied by a third of the radius.

Ex. 842. Show that the prismatoid formula can be used for finding the volume of a sphere.

Ex. 843. Find the altitude of a zone equivalent to a great circle.

Ex. 844. Find the area of a spherical pentagon whose angles are 122°, 128°, 131°, 160°, 161°, if the surface of the sphere is 150 square feet.

Construct a spherical surface with given radius:

Ex. 845. Passing through a given point and tangent to two given planes.

Ex. 846. Passing through a given point and tangent to two given spheres.

Ex. 847. Passing through a given point and tangent to a given plane and a given sphere.

Ex. 848. Tangent to three given planes.

Ex. 849. Tangent to three given spheres.

Ex. 850. Tangent to two given planes and a given sphere.

Ex. 851. Tangent to two given spheres and a given plane.

Ex. 852. Through a given point to pass a plane tangent to a given circular cylinder.

Ex. 853. Through a given point to pass a plane tangent to a given circular cone.

Ex. 854. Find the radius and the surface of a sphere whose volume is one cubic yard.

Ex. 855. Find the centre of a sphere whose surface passes through three given points, and touches a given plane.

Ex. 856. Find the centre of a sphere whose surface touches two given planes, and passes through two given points which lie between the planes.

Ex. 857. The volume of a sphere is two thirds the volume of the circumscribed circular cylinder, and its surface is two thirds the total surface of the cylinder.

Ex. 858. Given a sphere, a cylinder circumscribed about the sphere, and a cone of two nappes inscribed in the cylinder. If any two planes are drawn perpendicular to the axis of the three figures, the spherical segment between the planes is equivalent to the difference between the corresponding cylindrical and conic segments.

Ex. 859. A sphere 12 inches in diameter has an auger hole 3 inches in diameter through its centre. Find the remaining volume.

Ex. 860. Find the area of a solid generated by an equilateral triangle turning about one of its sides, if the length of the side is a.

Ex. 861. Compare the volumes of the solids generated by a rectangle turning successively about two adjacent sides, the lengths of these sides being a and b.

Ex. 862. An equilateral triangle revolves about one of its altitudes. Compare the lateral area of the cone generated by the triangle and the surface of the sphere generated by the inscribed circle.

Ex. 863. An equilateral triangle revolves about one of its altitudes. Compare the volumes of the solids generated by the triangle, the inscribed circle, and the circumscribed circle.

Ex. 864. The perpendicular let fall from the point of intersection of the medians of a given triangle upon any plane not cutting the triangle is equal to one third the sum of the perpendiculars from the vertices of the triangle upon the same plane.

Ex. 865. The perpendicular from the centre of gravity of a tetrahedron to a plane not cutting the tetrahedron is equal to one fourth the sum of the perpendiculars from the vertices of the tetrahedron to the plane.

BOOK IX.

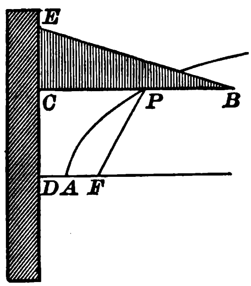
CONIC SECTIONS.

THE PARABOLA.

850. DEF. A **parabola** is a curve which is the locus of a point that moves in a plane so that its distance from a fixed point in the plane is always equal to its distance from a fixed line in the plane.

851. DEF. The fixed point is called the **focus**; and the fixed line, the **directrix**.

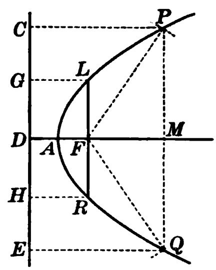
852. A parabola may be described by the continuous motion of a point, as follows:



Place a ruler so that one of its edges shall coincide with the directrix DE. Then place a right triangle with its base edge in contact with the edge of the ruler. Fasten one end of a string, whose length is equal to the other edge BC, to the point B, and the other end to a pin fixed at the focus F. Then slide the triangle BCE along the directrix, keeping the string tightly pressed against the ruler by the point of a pencil P. The point P will describe a parabola; for during the motion we always have PF equal to PC.

Proposition I. Problem.

853. *To construct a parabola by points, having given its focus and its directrix.*



Let F be the focus, and CDE the directrix.

Draw $FD \perp$ to CE, meeting CE at D. Bisect FD at A.

Then A is a point of the curve. § 850

Through any point M in the line DF, to the right of A, draw a line $\parallel$ to CE.

With F as centre and DM as radius, draw arcs cutting this line at the points P and Q.

Then P and Q are points of the curve.

Proof. Draw PC and $QE \perp$ to CE.

Then $PC = DM$, and $QE = DM$, § 180

and $DM = PF = QF$. Const.

$\therefore PC = PF$, and $QE = QF$. Ax. 1

Therefore, P and Q are points of the curve. § 850

In this way any number of points may be found; and a continuous curve drawn through the points thus determined is the parabola whose focus is F and directrix CDE. Q. E. F.

854. Def. The point A is called the **vertex** of the curve. The line DF produced indefinitely in both directions is called the **axis** of the curve.

855. Def. The line FP, joining the focus to any point P on the curve, is called the **focal radius** of P.

856. Def. The distance AM is called the **abscissa**, and the distance PM the **ordinate**, of the point P.

857. Def. The double ordinate LR, through the focus, is called the **latus rectum** or **parameter**.

858. Cor. 1. *The parabola is symmetrical with respect to its axis.* § 210

For $FP = FQ$ (Const.), and, therefore, $PM = QM$. § 149

859. Cor. 2. *The curve lies entirely on one side of the perpendicular to the axis erected at the vertex; namely, on the same side as the focus.*

For any point on the other side of this perpendicular is obviously nearer to the directrix than to the focus.

860. Cor. 3. *The parabola is not a closed curve.*

For any point on the axis of the curve to the right of F is evidently nearer to the focus than to the directrix. Hence, the parabola QAP cannot cross the axis to the right of F.

861. Cor. 4. *The latus rectum is equal to* $4\,AF$.

For, if LG is drawn $\perp$ to CDE,

then $LF = LG$, and $LG = DF$. § 850

$$\therefore LF = DF = 2\,AF.$$

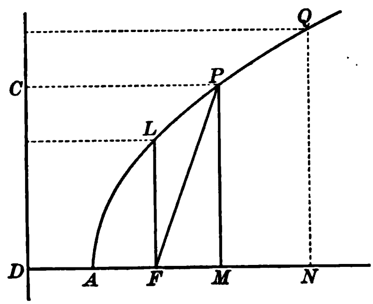
Similarly, $RF = DF = 2\,AF$.

Therefore, $LR = 4\,AF$.

Note. In the following propositions, the focus will be denoted by F, the vertex by A, and the point where the axis meets the directrix by D.

Proposition II. Theorem.

862. *The ordinate of any point of a parabola is the mean proportional between the latus rectum and the abscissa.*



Let P be any point, AM its abscissa, PM its ordinate.

To prove that $\overline{PM}^2 = 4\,AF \times AM.$

Proof. $\overline{PM}^2 = \overline{FP}^2 - \overline{FM}^2 = \overline{DM}^2 - \overline{FM}^2$ § 850

$= (DM - FM)(DM + FM)$

$= DF(DF + FM + FM)$

$= 2\,AF(2\,AF + 2\,FM)$

$= 2\,AF(2\,AM).$

Hence, $\overline{PM}^2 = 4\,AF \times AM.$

Q. E. D.

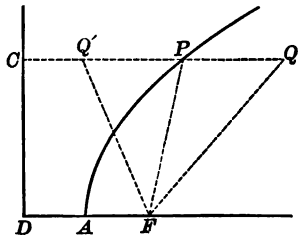
863. Cor. 1. *The greater the abscissa of a point, the greater the ordinate.*

864. Cor. 2. *The squares of any two ordinates are as the abscissas.*

For $$\frac{\overline{PM}^2}{\overline{QN}^2} = \frac{4\,AF \times AM}{4\,AF \times AN} = \frac{AM}{AN}.$$

Proposition III. Theorem.

865. *Every point within the parabola is nearer to the focus than to the directrix; and every point without the parabola is farther from the focus than from the directrix.*



1. Let Q be a point within the parabola. Draw QC perpendicular to the directrix, cutting the curve at P. Draw QF and PF.

To prove that $QF < QC$.

Proof. In the $\triangle QPF$, $QF < QP + PF$. § 138

$$\therefore QF < QP + PC.$$

$$\therefore QF < QC.$$

2. Let Q′ be a point without the curve. Draw Q′F.

To prove that $Q'F > Q'C$.

Proof. In the $\triangle Q'FP$, $Q'F > PF - PQ'$. § 138

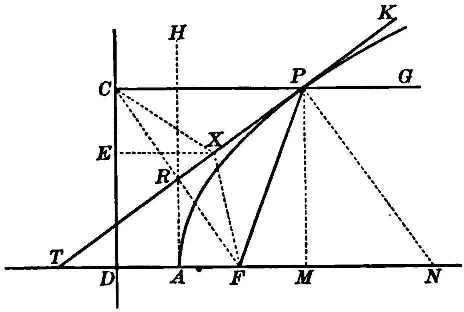
$$\therefore Q'F > PC - PQ' > Q'C.$$ Q. E. D.

866. Cor. *A point is within or without a parabola according as its distance from the focus is less than, or greater than, its distance from the directrix.*

867. Def. A straight line which touches, but does not cut, a parabola, is called a **tangent** to the parabola. The point where it touches the parabola is called the **point of contact.**

PROPOSITION IV. THEOREM.

868. *If a line PT is drawn from any point P of the curve, bisecting the angle between PF and the perpendicular from P to the directrix, every point of the line PT, except P, is without the curve.*



Let PC be the perpendicular from P to the directrix, the angle FPT equal the angle CPT, and let X be any other point in PT except P.

To prove that X is without the curve.

Proof. Draw $XE \perp$ to the directrix, and draw CX, FX, CF.

Let CF meet PT at R.

In the isos. $\triangle PCF$, $CR = RF$, § 149

and $CX = FX$. § 160

But $EX < CX$. § 97

$\therefore EX < FX$.

Hence, X is without the curve. § 866

Q.E.D.

869. COR. 1. *The bisector of the angle between PF and the perpendicular from P to the directrix is tangent to the curve at P.* § 867

870. COR. 2. *PT bisects FC, and is perpendicular to FC.*

871. Cor. 3. *Since the angles* FPT *and* FTP *are equal,* FT *equals* FP. § 147

872. Cor. 4. *The tangent at* A *is perpendicular to the axis.*

For it bisects the straight angle FAD.

873. Cor. 5. *The tangent at* A *is the locus of the foot of the perpendicular dropped from the focus to any tangent.*

Since $FR = RC$, and $FA = AD$, R is in AH. § 189

874. Def. The line PN drawn through P perpendicular to the tangent PT is called the **normal** at P.

875. Def. If the ordinate of P meets the axis in M, and the tangent and normal at P meet the axis in T and N respectively, then MT is the **subtangent** and MN the **subnormal**.

876. Cor. 6. *The subtangent is bisected by the vertex.*

For $\quad FT = FP$ (§ 871), and $FP = DM$. § 850

Hence, $\quad FT = DM$; also $AF = AD$.

Therefore, $\quad FT - AF = DM - AD$. Ax. 3

That is, $\quad TA = AM$.

877. Cor. 7. *The subnormal is equal to half the latus rectum.*

For $\quad CP = FN$, and $CP = DM$. § 180

$\therefore FN = DM$ (Ax. 1); that is, $FM + MN = DF + FM$.

Therefore, $\quad MN = DF$. Ax. 3

878. Cor. 8. *The normal bisects the angle between* FP *and* CP *produced; that is, bisects the angle* FPG.

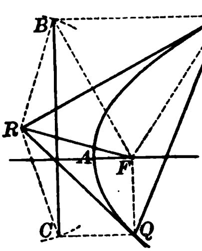
For $\angle NPT = \angle NPK$, and $\angle FPT = \angle TPC = \angle GPK$.

Hence, $\quad \angle NPF = \angle NPG$. Ax. 3

879. Cor. 9. *The circle with* F *as centre and* FP *as radius passes through* T *and* N.

PROPOSITION V. PROBLEM.

880. *To draw a tangent to a parabola from an external point.*



Let R be any point external to the parabola QAP.

With R as centre and RF as radius, draw arcs cutting the directrix at the points B, C. Through B and C draw lines parallel to the axis, meeting the parabola in P, Q, respectively. Draw RP, RQ.

Then RP and RQ are tangents to the curve.

Proof. By construction, $RB = RF$, and $PB = PF$. § 850

Hence, $\angle RPB = \angle RPF$. § 150

Therefore, RP is the tangent at P. § 869

For like reason, RQ is the tangent at Q. Q.E.F.

881. COR. *Two tangents can always be drawn to a parabola from an external point.*

For R is without the curve and nearer to the directrix than to the focus (§ 865); therefore, the circle with R as centre and RF as radius must always cut the directrix in two points.

882. DEF. The line joining the points of contact P and Q is called the **chord of contact** for the tangents drawn from R.

PROPOSITION VI. THEOREM.

883. *The line joining the focus to the intersection of two tangents makes equal angles with the focal radii drawn to the points of contact.*

Let the tangents drawn at P and Q meet in R.

To prove that $\angle RFP = \angle RFQ$.

Proof. Draw the ⊥s PB, QC to the directrix, and draw RB, RC, RF, FB.

	$\triangle RFP = \triangle RBP$.	§ 150
For	$PB = PF$,	§ 850
and	$RB = RF$.	§§ 870, 160
	$\therefore \angle RFP = \angle RBP$.	§ 128
Similarly,	$\angle RFQ = \angle RCQ$.	
Now	$\angle RBP = 90° + \angle RBC$,	
and	$\angle RCQ = 90° + \angle RCB$;	
and since	$RB = RF$, and $RC = RF$,	
therefore,	$RB = RC$.	Ax. 1
Hence,	$\angle RBC = \angle RCB$.	§ 145
Therefore,	$\angle RBP = \angle RCQ$,	
and	$\angle RFP = \angle RFQ$.	Q. E. D.

884. COR. *The tangents drawn through the ends of a focal chord meet in the directrix.*

For, if the chord of contact PQ passes through F, then PFQ is a straight line.

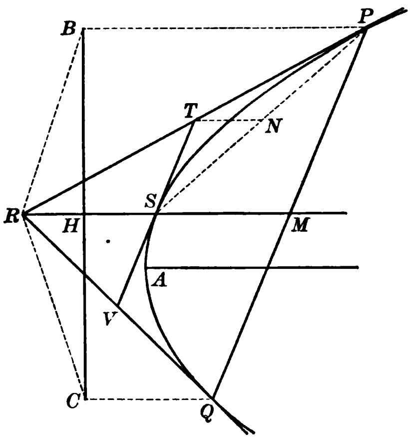
Hence, $\angle RFP + \angle RFQ = 180°$,

and $\angle RFP = \angle RFQ = 90°$.

Therefore, $\angle RBP = \angle RCQ = 90°$.

Proposition VII. Theorem.

885. *If two tangents RP and RQ are drawn from a point R to a parabola, the line drawn through R parallel to the axis bisects the chord of contact.*



Let the tangents drawn from R meet the curve in P, Q, and let the line through R parallel to the axis meet the directrix in H, the curve in S, and the chord of contact in M.

To prove that $PM = QM$.

Proof. Drop the ⊥s PB and QC to the directrix,

and draw RB, RC.

RH is $\perp$ to BC, § 107

$RB = RC$. § 883

Hence, $BH = CH$. § 149

Now PB, QC, and MR are parallel. § 104

$\therefore PM = QM$. § 187

Q. E. D.

PROPOSITION VIII. THEOREM.

886. *If two tangents RP, RQ are drawn from a point R to a parabola, and through R a line parallel to the axis is drawn, meeting the curve in S, then the tangent at S is parallel to the chord of contact.*

Let the tangent at S meet the tangents PR, QR in T, V, respectively.

To prove that TV *is* $\parallel$ *to* PQ.

Proof. Draw $TN \parallel$ to SM, and let it meet SP in N.

Then	$PN = NS.$	§ 885
Hence,	$PT = TR.$	§ 188
Similarly,	$QV = VR.$	
Therefore,	TV is $\parallel$ to PQ.	§ 189

Q. E. D.

887. COR. 1. *The line RM is the locus of the middle points of all chords drawn parallel to the tangent at S.*

For, if we suppose R to move along RM towards the curve, then since the point S and the direction of the tangent TV remain fixed, the chord PQ will remain parallel to TV, while its middle point M will move along MR towards S; finally, R, M, P, and Q will all coincide at S.

888. DEF. The locus of the middle points of a system of parallel chords in a parabola is called a **diameter**. The parallel chords are called the **ordinates** of the diameter.

889. COR. 2. *The diameters of a parabola are parallel to its axis; and conversely, every straight line parallel to the axis is a diameter; that is, bisects a system of parallel chords.*

890. COR. 3. *Tangents drawn through the ends of an ordinate intersect in the diameter corresponding to that ordinate.*

891. Cor. 4. *The portion of a diameter contained between any ordinate and the intersection of the tangents drawn through the ends of the ordinate is bisected by the curve.*

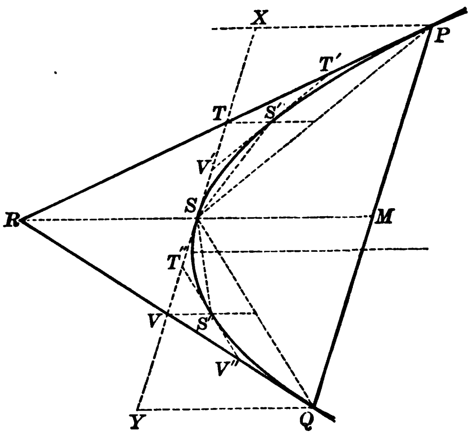
For the point S is the middle point of RM. § 188

892. Cor. 5. *The part of a tangent parallel to a chord contained between the two tangents drawn through the ends of the chord is bisected by the diameter of the chord at the point of contact.*

For the point S is also the middle point of the tangent TV.

Proposition IX. Theorem.

893. *The area of a parabolic segment made by a chord is two thirds the area of the triangle formed by the chord and the tangents drawn through the ends of the chord.*



Let PQ be any chord, and let the tangents at P and Q meet in R.

To prove that segment $PSQ \approx \frac{2}{3} \triangle PRQ$.

Proof. Draw the diameter RM, meeting the curve at S, and at S draw a tangent meeting PR in T and QR in V.

Draw SP, SQ.

Since $PT = TR$, and $QV = VR$, § 886

TV is $\parallel$ to PQ, § 189

and $PQ = 2 \times TV$. § 189

$\therefore \triangle PQS \backsimeq 2 \triangle TVR$. § 405

If now we draw through T, V, the diameters TS', VS'', and then draw through S', S'', the tangents $T'S'V'$, $T''S''V''$, we can prove in the same way that

$$\triangle PSS' \backsimeq 2 \triangle T'V'T,$$

and

$$\triangle QSS'' \backsimeq 2 \triangle T''V''V.$$

If we continue to form new triangles by drawing diameters through the points T', V', T'', V'', and tangents at the points where these diameters meet the curve, we can prove that each interior triangle formed by joining a point of contact to the extremities of a chord is twice as large as the exterior triangle formed by the tangents through these points, and hence that the sum of all the interior triangles is equal to twice the sum of the corresponding exterior triangles.

Now if we suppose the process to be continued indefinitely, then the limit of the sum of the interior triangles will be the segment PQS, and the limit of the sum of the exterior triangles will be the figure contained between the tangents PR, QR, and the curve.

But the sum of the interior triangles will always be equal to twice the sum of the exterior triangles; that is, to $\frac{2}{3}$ of the whole area, or $\frac{2}{3}$ the $\triangle PQR$.

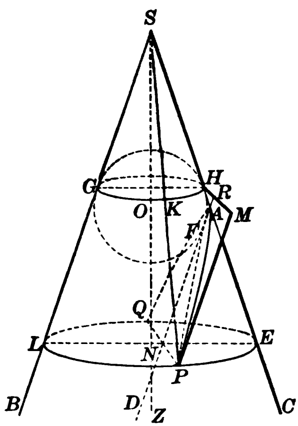
$\therefore$ segment $PQS \backsimeq \frac{2}{3} \triangle PQR$. § 284

Q. E. D.

894. Cor. *If through* P *and* Q *lines are drawn parallel to* SM, *meeting the tangent* TV *produced in the points* X *and* Y, *then the segment* $PQS \backsimeq \frac{4}{3} \square PQYX$.

Proposition X. Theorem.

895. *The section of a right circular cone made by a plane parallel to one, and only one, element of the surface is a parabola.*



Let SB be any element of the cone whose axis is SZ, and let QAP be the section of the cone made by a plane perpendicular to the plane BSZ and parallel to SB.

To prove that the curve PAQ is a parabola.

Proof. Let SC be the second element in which the plane BSZ cuts the cone, and let RAD be the intersection of the planes BSZ and PAQ.

Draw the ⊙ O tangent to the lines SB, SC, RD, and let G, H, F be the points of contact, respectively.

Revolve BSC and the ⊙ O about the axis SZ, the plane PAQ remaining fixed. The ⊙ O will generate a sphere which will touch the cone in the ⊙ GKH, and the plane PAQ at the point F.

Since SZ is $\perp$ to GH, SZ is $\perp$ to the plane GKH. § 501

Hence, the plane BSC is $\perp$ to the plane GKH. § 554

Let the plane of the $\odot$ GKH intersect the plane of the curve PAQ in the straight line MR; then will MR be $\perp$ to the plane BSC (§ 556), and therefore $\perp$ to DR.

Take any point P in the curve PAQ, and draw SP meeting the $\odot$ GH in K; draw FP, and draw PM $\perp$ to RM.

Pass a plane through P $\perp$ to the axis of the cone. Let it cut the cone in the $\odot$ $EPLQ$, and the plane of the curve PAQ in the line PNQ.

PN is $\perp$ to the plane BSC (§ 556), and therefore $\perp$ to DR.

Since PF and PK are tangents to the sphere O, they are tangents to the circle of the sphere made by a plane passing through the points P, F, K, and are therefore equal. § 261

That is, $PF = PK$.

But $PK = LG$. § 716

$\therefore PF = LG$. Ax. 1

Now LG and PM are each $\parallel$ to NR;

hence, LG is $\parallel$ to PM. § 521

The planes GKH and LPE are parallel. § 527

$\therefore LG = PM$. § 529

But $PF = LG$.

$\therefore PF = PM$. Ax. 1

That is, any point P on the curve PAQ is equidistant from a fixed point F and a fixed line RM in its plane.

Therefore, the curve PAQ is a parabola. § 850

Q. E. D.

EXERCISES.

Ex. 866. If the abscissa of a point is equal to its ordinate, each is equal to the latus rectum.

Ex. 867. If a secant PP' meets the directrix at H, then HF is the bisector of the exterior angle between the focal radii FP and FP'.

A straight line that cuts the curve is called a secant.

Ex. 868. To draw a tangent and a normal at a given point of a parabola.

Ex. 869. To draw a tangent to a parabola parallel to a given line.

Ex. 870. The tangents at the ends of the latus rectum meet at D.

Ex. 871. The latus rectum is the shortest focal chord.

Ex. 872. The tangent at any point meets the directrix and the latus rectum produced at points equally distant from the focus.

Ex. 873. The circle whose diameter is FP touches the tangent at A.

Ex. 874. The directrix touches the circle that has any focal chord for diameter.

Ex. 875. Given two points and the directrix, to find the focus.

Ex. 876. The perpendicular FC bisects TP. (See figure, page 414.)

Ex. 877. Given the focus and the axis, to describe a parabola which shall touch a given straight line.

Ex. 878. If PN is any normal, and the triangle PNF is equilateral, then PF is equal to the latus rectum.

Ex. 879. Given a parabola, to find the directrix, axis, and focus.

Ex. 880. To find the locus of the centre of a circle which passes through a given point and touches a given straight line.

Ex. 881. Given the axis, a tangent, and the point of contact, to find the focus and directrix.

Ex. 882. Given two points and the focus, to find the directrix.

Ex. 883. The triangles formed by the two tangents from any point and the focal radii to the points of contact are similar.

Ex. 884. If a diameter of a parabola is cut by a chord and the tangent at either end of the chord, the segments of the diameter between the tangent and the chord made by the curve are in the same ratio as the segments of the chord.

THE ELLIPSE.

896. Def. An **ellipse** is a curve which is the locus of a point that moves in a plane so that the sum of its distances from two fixed points in the plane is constant.

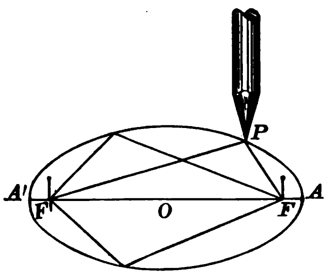
897. Def. The fixed points are called the **foci**, and the straight lines which join a point of the curve to the foci are called the **focal radii** of that point.

898. The constant sum of the focal radii is denoted by $2a$, and the distance between the foci by $2c$.

899. Def. The ratio $c:a$ is called the **eccentricity**, and is denoted by e. Therefore, $c = ae$.

900. Cor. *$2a$ must be greater than $2c$* (§ 138); *hence, e must be less than* 1.

901. The curve may be described by the continuous motion of a point, as follows:

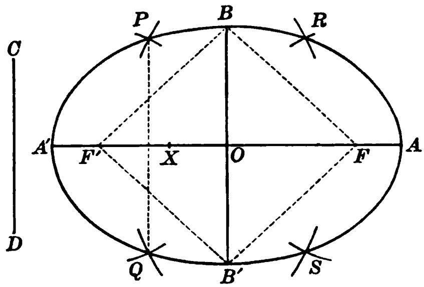


Fasten the ends of a string whose length is $2a$ at the foci F and F'. Trace a curve with the point P of a pencil pressed against the string so as to keep it stretched. The curve thus traced will be an ellipse whose foci are F and F', and the constant sum of whose focal radii is $FP + PF'$.

The curve is a closed curve extending around both foci; if A and A' are the points in which the curve cuts FF' produced, then AA' equals the length of the string.

Proposition XI. Problem.

902. *To construct an ellipse by points, having given the foci and the constant sum* $2a$.



Let F and F' be the foci, and CD equal a.

Through the foci F, F' draw a straight line; bisect FF' at O. Lay off OA' equal to OA equal to CD.

Then A and A' are two points of the curve.

Proof. From the construction, $AA' = 2a$, and $AF = A'F'$.

Therefore, $AF + AF' = A'F + A'F' = AA' = 2a$,

and $A'F + A'F' = A'F + AF = AA' = 2a$.

To locate other points, mark any point X between F and F'. Describe arcs with F as centre and AX as radius; also other arcs with F' as centre and $A'X$ as radius; let these arcs cut in P and Q.

Then P and Q are two points of the curve.

This follows at once from the construction and § 896.

By describing the same arcs *with the foci interchanged*, two more points R, S may be found.

By assuming other points between F and F', and proceeding in the same way, any number of points may be found.

The curve passing through all the points is an ellipse having F and F' for foci, and $2a$ for the constant sum of focal radii.

Q. E. F.

903. Cor. 1. *By describing arcs from the foci with the same radius* OA, *we obtain two points* B, B' *of the curve which are equidistant from the foci. Therefore the line* BB' *is perpendicular to* AA' *and passes through* O. § 161

904. Def. The point O is called the **centre.** The line AA' is called the **major axis**; its ends A, A' are called the **vertices** of the curve. The line BB' is called the **minor axis.** The length of the minor axis is denoted by $2b$.

905. Cor. 2. *The major axis is bisected at* O, *and is equal to the constant sum* $2a$.

906. Cor. 3. *The minor axis is also bisected at* O. § 161

Therefore, $OB = OB' = b$.

907. Cor. 4. *The values of* a, b, c *are so related that*

$$a^2 = b^2 + c^2.$$

For, in the rt. $\triangle BOF$,

$$\overline{BF}^2 = \overline{OB}^2 + \overline{OF}^2. \qquad \text{§ 371}$$

908. Cor. 5. *The ellipse is symmetrical with respect to its major axis.*

For the axis AA' bisects PQ at right angles. § 161

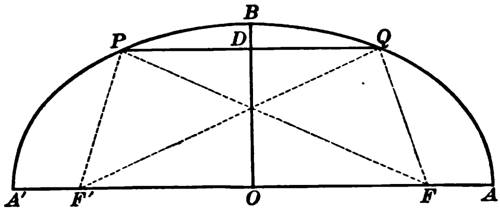
909. Def. The distance of a point of the curve from the minor axis is called the **abscissa** of the point, and its distance from the major axis is called the **ordinate** of the point.

910. Def. The double ordinate through the focus is called the **latus rectum** or **parameter**.

Note. In the following propositions F and F' denote the foci of the ellipse, O the centre, AA' the major axis, and BB' the minor axis.

Proposition XII. Theorem.

911. *An ellipse is symmetrical with respect to its minor axis.*



Let P be a point of the curve, PDQ be perpendicular to OB, meeting OB in D, and let DQ equal DP.

To prove that Q is also a point of the curve.

Proof. Join P and Q to the foci F, F'.

Revolve $ODQF$ about OD; F will fall on F', and Q on P.

Therefore,	$QF = PF'$,	
and	$\angle PQF = \angle QPF'$.	
Therefore,	$\triangle PQF = \triangle QPF'$,	§ 143
and	$QF' = PF$.	§ 128
Hence,	$QF + QF' = PF + PF'$.	Ax. 2
But	$PF + PF' = 2a$.	Hyp.
Therefore,	$QF + QF' = 2a$.	Ax. 1
Therefore,	Q is a point of the curve.	§ 896

Q.E.D.

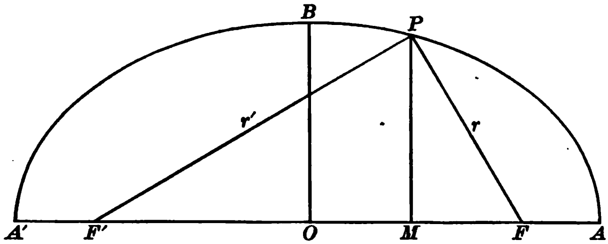
912. Def. Every chord that passes through the centre of an ellipse is called a **diameter**.

913. Cor. 1. *From* §§ 908, 911 *it follows that an ellipse consists of four equal quadrantal arcs symmetrically placed about the centre.* § 213

914. Cor. 2. *Every diameter is bisected at the centre.* § 209

Proposition XIII. Theorem.

915. *If d denotes the abscissa of a point of an ellipse, r and r′ its focal radii, then* $r' = a + ed$, $r = a - ed$.



Let P be any point of an ellipse, PM perpendicular to AA′, d equal OM, r equal PF, r′ equal PF′.

To prove that $r' = a + ed,\ r = a - ed.$

Proof. From the rt. △ FPM and $F'PM$,

$$r^2 = \overline{PM}^2 + \overline{FM}^2,$$

and $$r'^2 = \overline{PM}^2 + \overline{F'M}^2. \qquad \S\ 371$$

Therefore, $r'^2 - r^2 = \overline{F'M}^2 - \overline{FM}^2.$ Ax. 3

Or $(r' + r)(r' - r) = (F'M + FM)(F'M - FM).$

Now $r' + r = 2a$, and $F'M + FM = 2c.$

Also, $F'M - FM = OF' + OM - FM = 2\,OM = 2d.$

Hence, $2a(r' - r) = 4cd.$

$$\therefore r' - r = \frac{2cd}{a} = 2ed.$$

From $r' + r = 2a$, and $r' - r = 2ed$,

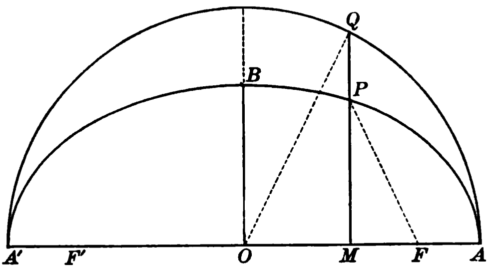
$$2r' = 2(a + ed),\ \text{and}\ 2r = 2(a - ed).$$

Therefore, $r' = a + ed$, and $r = a - ed$. Q.E.D.

916. Def. The circle described upon the major axis of an ellipse as a diameter is called the **auxiliary circle.** The points where a line perpendicular to the major axis meets the ellipse and its auxiliary circle are called **corresponding points.**

Proposition XIV. Theorem.

917. *The ordinates of two corresponding points in an ellipse and its auxiliary circle are in the ratio $b : a$.*



Let P be a point of the ellipse, Q the corresponding point of the auxiliary circle, and QP meet AA′ at M.

To prove that $PM : QM = b : a.$

Proof. Let $OM = d;$

then $\overline{QM}^2 = a^2 - d^2.$ § 371

Now $\overline{PM}^2 = \overline{PF}^2 - \overline{FM}^2 = (a - ed)^2 - (c - d)^2$ § 915

$$= a^2 - 2\,aed + e^2d^2 - c^2 + 2\,cd - d^2.$$

Or, since $c = ae,$ and $a^2 - c^2 = b^2,$ §§ 899, 907

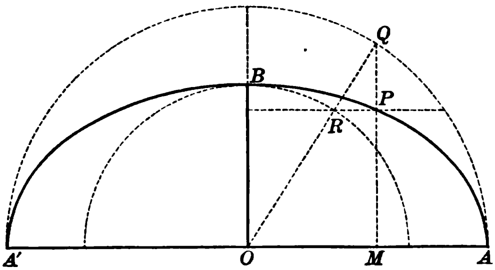
$$\overline{PM}^2 = b^2 - (1 - e^2)\,d^2 = \frac{b^2}{a^2}(a^2 - d^2).$$

Therefore, $\overline{PM}^2 : \overline{QM}^2 = b^2 : a^2.$

Or $PM : QM = b : a.$ Q.E.D.

PROPOSITION XV. PROBLEM.

918. *To construct an ellipse by points, having given its two axes.*



Let OA, OB be the given semi-axes, O the centre.

With O as centre, and OA, OB, respectively, as radii, describe circles.

From O draw any straight line meeting the larger circle at Q and the smaller circle at R.

Through Q draw a line ∥ to BO, and through R draw a line ∥ to OA.

Let these lines meet at P.

Then will P be a point of the required ellipse.

Proof. If QP meet AA' at M,

$PM : QM = OR : OQ$. § 343

But $OR = b$ and $OQ = a$.

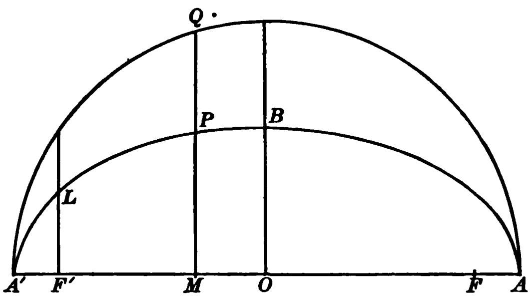
Therefore, $PM : QM = b : a$.

Therefore, P is a point of the ellipse. § 917

By drawing other lines through O, any number of points on the ellipse may be found; a smooth curve drawn through all the points will be the ellipse required. Q.E.F.

PROPOSITION XVI. THEOREM.

919. *The square of the ordinate of a point in an ellipse is to the product of the segments of the major axis made by the ordinate as* $b^2 : a^2$.



Let P, Q be corresponding points in the ellipse and auxiliary circle, respectively; let QP meet AA′ in M.

To prove that $\overline{PM}^2 : AM \times A'M = b^2 : a^2$.

Proof. $\overline{PM}^2 : \overline{QM}^2 = b^2 : a^2$. § 917

But $\overline{QM}^2 = AM \times A'M$. § 370

Therefore, $\overline{PM}^2 : AM \times A'M = b^2 : a^2$. Q.E.D.

920. COR. *The latus rectum is the third proportional to the major axis and the minor axis.*

For $\overline{LF'}^2 : AF' \times A'F' = b^2 : a^2$. § 919

Now $A'F' = a - c$, and $AF' = a + c$.

Therefore, $AF' \times A'F' = a^2 - c^2 = b^2$. § 907

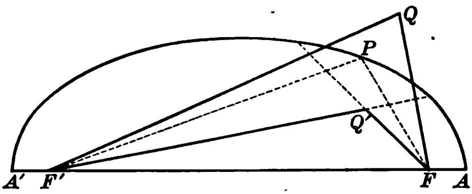
Hence, $\overline{LF'}^2 : b^2 = b^2 : a^2$,

and $LF' : b = b : a$.

Therefore, $2a : 2b = 2b : 2LF'$.

Proposition XVII. Theorem.

921. *The sum of the distances of any point from the foci of an ellipse is greater than or less than $2a$, according as the point is without or within the curve.*



1. **Let Q be a point without the curve.**

To prove that $QF + QF' > 2a$.

Proof. Let P be any point on the curve between QF and QF'. Draw PF and PF'.

Then $QF + QF' > PF + PF'$. § 100

But $PF + PF' = 2a$. § 896

Therefore, $QF + QF' > 2a$.

2. **Let Q' be a point within the curve.**

To prove that $Q'F + Q'F' < 2a$.

Proof. Let P be any point of the curve included between FQ' produced and $F'Q'$ produced.

Then $Q'F + Q'F' < PF + PF'$. § 100

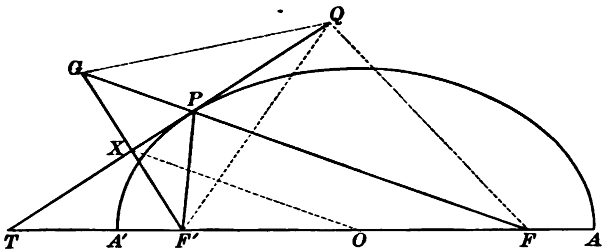
That is, $Q'F + Q'F' < 2a$. Q.E.D.

922. Cor. *Conversely, a point is without or within an ellipse according as the sum of its distances from the foci is greater than or less than $2a$.*

923. Def. A straight line which touches but does not cut an ellipse is called a **tangent** to the ellipse. The point where a tangent touches the ellipse is called the **point of contact.**

PROPOSITION XVIII. THEOREM.

924. *If through a point P of an ellipse a line is drawn bisecting the angle between one of the focal radii and the other produced, every point in this line except P is without the curve.*



Let PT bisect the angle F′PG between F′P and FP produced, and let Q be any point in PT except P.

To prove that Q is without the curve.

Proof. Upon FP produced take PG equal to PF'.

Draw GF', QF, QF', QG.

Then	$QG + QF > GF.$	§ 138
Now	$\triangle GPQ = \triangle F'PQ.$	§ 143
Therefore,	$QG = QF'.$	§ 128
Also	$GF = 2a.$	Const.
Therefore,	$QF' + QF > 2a.$	
Therefore,	Q is without the curve.	§ 922

Q. E. D.

925. COR. 1. *The bisector of the angle between one of the focal radii from any point P and the other produced through P is a tangent to the curve at P.* § 923

926. COR. 2. *The tangent to an ellipse at any point bisects the angle between one focal radius and the other produced.*

927. COR. 3. *If GF' cuts PT at X, then $GX = F'X$, and PT is perpendicular to GF'.* § 161

928. COR. 4. *The locus of the foot of a perpendicular from the focus of an ellipse to a tangent is the auxiliary circle.*

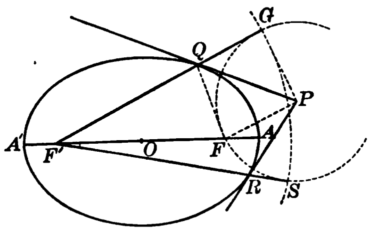
For $F'X = GX$, and $F'O = OF$.

Therefore, $OX = \frac{1}{2} FG = \frac{1}{2}(2a) = a$. § 189

Therefore, the point X lies in the auxiliary circle.

PROPOSITION XIX. PROBLEM.

929. *To draw a tangent to an ellipse from an external point.*



Let the arcs drawn with P as centre and PF as radius, and with F' as centre and $2a$ as radius intersect in G and S.

Draw GF' and SF', cutting the curve in Q and R, respectively.

Draw QP and RP, and they will be the tangents required.

Proof. By construction, $PG = PF$, and $QG = QF$. § 896

$\therefore \triangle PQG = \triangle PQF$. § 150

$\therefore \angle PQG = \angle PQF$. § 128

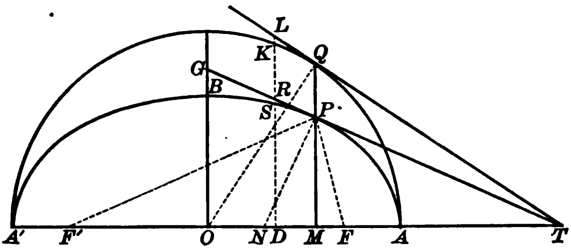
Therefore, PQ is the tangent at Q. § 925

For like reason PR is the tangent at R. Q.E.F.

930. COR. *Two tangents may always be drawn to an ellipse from an external point.*

Proposition XX. Theorem.

931. *The tangents drawn at two corresponding points of an ellipse and its auxiliary circle cut the major axis produced at the same point.*



Let the tangent to the auxiliary circle at Q cut the major axis produced at T, and let the ordinate QM of the circle meet the ellipse at P. Draw TG through P.

To prove that TG is the tangent to the ellipse at P.

Proof. Through S, any point in the ellipse except P, draw $SD \perp$ to AA'; and let DS produced cut TG in R, the auxiliary circle in K, and the tangent at Q in L.

Then $\quad RD : PM = DT : MT = LD : QM,$ §§ 356, 351

or $\quad RD : LD = PM : QM.$ § 330

But $\quad PM : QM = b : a.$ § 917

$\therefore RD : LD = b : a.$ Ax. 1

Again, $\quad SD : KD = b : a.$ § 917

$\therefore RD : LD = SD : KD.$ Ax. 1

But $\quad LD > KD.$ Ax. 8

$\therefore RD > SD.$

$\therefore R$ is without the ellipse.

Hence, PT is the tangent at P. § 923

Q.E.D.

932. Cor. 1. $OT \times OM = a^2$. § 367

933. Def. The straight line PN drawn through the point of contact of a tangent, perpendicular to the tangent, is called the **normal**.

934. Def. MT is called the **subtangent**, MN the **subnormal**.

935. Cor. 2. *The normal bisects the angle between the focal radii of the point of contact.*

For $\angle TPN = \angle GPN = 90°$. § 933

But $\angle TPF = \angle GPF'$. § 926

$\therefore \angle FPN = \angle F'PN$. Ax. 3

Hence, a ray of light issuing from F will be reflected to F'.

936. Cor. 3. *If d denotes the abscissa of the point of contact, the distances measured on the major axis from the centre to the tangent and the normal are $\frac{a^2}{d}$ and e^2d, respectively.*

Since $OM = d$, and $OT \times OM = a^2$, § 932

therefore, $$OT = \frac{a^2}{d}.$$

Since $OM \times MT = \overline{QM}^2$, § 367

and $MN \times MT = \overline{PM}^2$,

therefore, $$\frac{OM}{MN} = \frac{\overline{QM}^2}{\overline{PM}^2} = \frac{a^2}{b^2}.$$

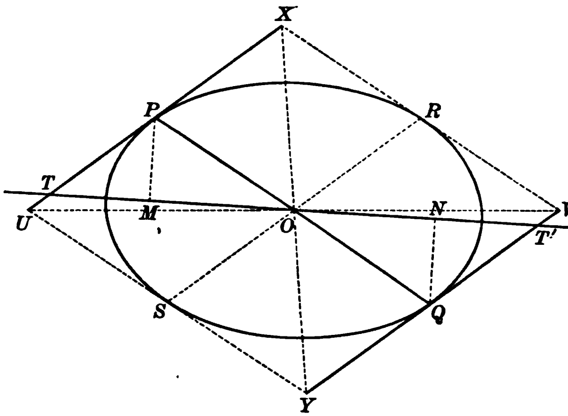
Therefore, $$\frac{OM - MN}{OM} = \frac{a^2 - b^2}{a^2} = \frac{c^2}{a^2} = e^2.$$ § 333

That is, $$\frac{ON}{OM} = e^2.$$

Hence, $ON = e^2 \times OM = e^2d$.

Proposition XXI. Theorem.

937. *The tangents drawn at the ends of any diameter are parallel to each other.*



Let POQ be any diameter, PT and QT′ the tangents at P, Q, respectively, meeting the major axis at T, T′.

To prove that PT is ∥ to QT'.

Proof. Draw the ordinates PM, QN.

Then $\triangle OPM = \triangle OQN$ (§ 141), and $OM = ON$. § 128

But $OT = \dfrac{a^2}{OM}$, and $OT' = \dfrac{a^2}{ON}$ (§ 936). $\therefore OT = OT'$.

Therefore, $\triangle OPT = \triangle OQT'$, § 143

and $\angle OPT = \angle OQT'$. § 128

Hence, PT is ∥ to QT'. § 111

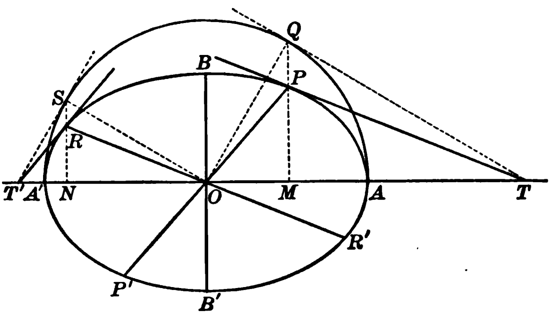
Q. E. D.

938. Def. One diameter is **conjugate** to another, if the first is parallel to the tangents at the extremities of the second.

Thus, if ROS is ∥ to PT, RS is conjugate to PQ.

Proposition XXII. Theorem.

939. *If one diameter is conjugate to a second, the second is conjugate to the first.*



Let the diameter POP′ be parallel to the tangent RT′.

To prove that ROR' is parallel to the tangent PT.

Proof. Draw the ordinates PM and RN, and produce them to meet the auxiliary circle in Q and S, respectively.

Draw OP, OQ, OS; and draw the tangents QT, ST'.

Now, since OP is ∥ to RT',

	the $\triangle$ OMP and $T'NR$ are similar.	§ 354
	$\therefore T'N : OM = NR : MP.$	§ 351
But	$NR : NS = MP : MQ,$	§ 917
or	$NR : MP = NS : MQ.$	§ 330
	$\therefore T'N : OM = NS : MQ.$	Ax. 1
Hence,	$\triangle$ $T'NS$ and OMQ are similar.	§ 357
	$\therefore \angle NT'S = \angle MOQ.$	§ 351
	$\therefore T'S$ is ∥ to OQ.	§ 114
Hence,	$\angle QOS = \angle OST' = 90°.$	§ 254
	$\therefore SO$ is ∥ to QT.	§ 104

$\therefore \triangle$ *SNO* and *QMT* are similar. § 354

$\therefore ON : TM = NS : MQ,$ § 351

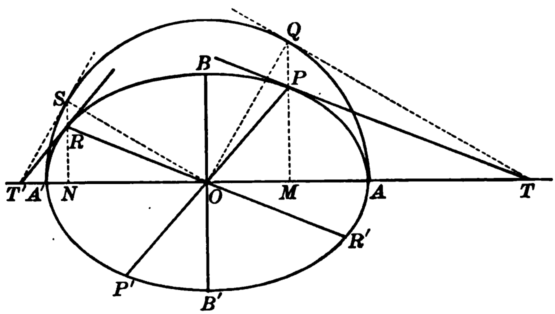
$= NR : MP.$ § 917

$\therefore \triangle$ *ONR* and *TMP* are similar. § 357

$\therefore OR$ is $\parallel$ to PT. § 114

$\therefore RR'$ is conjugate to PP'. § 938

Q. E. D.



940. Cor. 1. *Angle QOS is a right angle.*

941. Cor. 2. $MP : ON = b : a$.

For $OS = OQ,$ § 217

and since $\angle NST' = \angle MQO,$ § 176

$\angle NSO = \angle MOQ.$ § 84

Hence, $\triangle NSO = \triangle MOQ.$ § 141

$\therefore ON = MQ.$ § 128

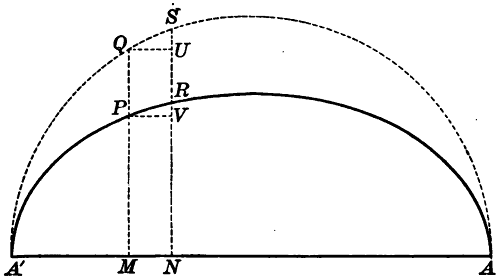
$\therefore MP : ON = MP : MQ.$

But $MP : MQ = b : a.$ § 917

Hence, $MP : ON = b : a.$ Ax. 1

PROPOSITION XXIII. THEOREM.

942. *The area of an ellipse is equal to πab.*



Let A′PRA be any semi-ellipse.

To prove that the area of twice A′PRA is equal to πab.

Proof. Let PM, RN be two ordinates of the ellipse, and let Q, S be the corresponding points on the auxiliary circle.

Draw PV, QU ∥ to the major axis, meeting NS in V, U.

Then the area of $\square\, PN = PM \times MN$, § 398

and the area of $\square\, QN = QM \times MN$. § 398

Therefore, $\dfrac{\square\, PN}{\square\, QN} = \dfrac{PM \times MN}{QM \times MN} = \dfrac{PM}{QM} = \dfrac{b}{a}$. § 917

The same relation will be true for all the rectangles that can be similarly drawn in the ellipse and auxiliary circle.

Hence, $\dfrac{\text{sum of } ⧠\text{s in ellipse}}{\text{sum of } ⧠\text{s in circle}} = \dfrac{b}{a}$. § 335

And this is true whatever be the number of the rectangles.

But the limit of the sum of the ⧠s in the ellipse is the area of the ellipse, and the limit of those in the ⊙ is the area of the ⊙.

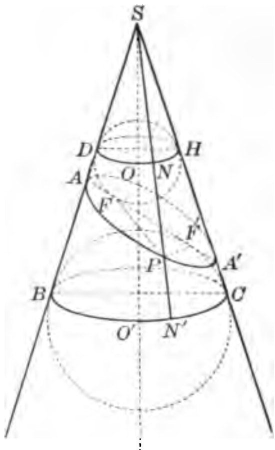
Therefore, $\dfrac{\text{area of ellipse}}{\text{area of circle}} = \dfrac{b}{a}$. § 285

Therefore, the area of the ellipse $= \dfrac{b}{a} \times \pi a^2 = \pi ab$. § 463

Q.E.D.

Proposition XXIV. Theorem.

943. *The section of a right circular cone made by a plane that cuts all the elements of the surface of the cone is an ellipse.*



Let APA′ be the curve traced on the surface of the cone SBC by a plane that cuts all the elements of the surface of the cone.

To prove that the curve APA' is an ellipse.

Proof. The plane passed through the axis of the cone $\perp$ to the secant plane APA' cuts the surface of the cone in the elements SB, SC, and the secant plane in the line AA'.

Describe the ⊙ O and O' tangent to SB, SC, AA'. Let the points of contact be D, H, F, and B, C, F', respectively.

Turn BSC and the ⊙ O, O' about the axis of the cone. The lines SB, SC will generate the surface of a right circular cone cut by the secant plane in the curve APA'; and the ⊙ O, O' will generate spheres which touch the cone in the ⊙ DNH, $BN'C$, and the secant plane in the points F, F'.

Let P be any point on the curve APA'. Draw PF, PF'; and draw SP, which touches the Ⓢ DH, BC at the points N, N', respectively.

Since PF and PN are tangent to the sphere O, they are tangent to the circle of the sphere made by a plane passing through P, F, and N.

Therefore, $PF = PN$. § 261

Likewise, $PF' = PN'$. § 261

Hence, $PF + PF' = PN + PN'$

$= NN'$, a constant quantity. § 716

Therefore, APA' is an ellipse with the points F and F' for foci, and AA' as $2a$. Q.E.D.

944. Cor. *If the secant plane is parallel to the base, the section is a circle.*

Ex. 885. The major axis is the longest chord that can be drawn in an ellipse.

Ex. 886. If the angle FBF' is a right angle, then $a^2 = 2b^2$.

Ex. 887. To draw a tangent and a normal at a given point of an ellipse.

Ex. 888. To draw a tangent to an ellipse parallel to a given straight line.

Ex. 889. Given the foci; to describe an ellipse touching a given straight line.

Ex. 890. Prove that $\overline{OF}^2 = OT \times ON$. (See figure, page 436.)

Ex. 891. Prove that $OM : ON = a^2 : c^2$. (See figure, page 436.)

Ex. 892. The minor axis is the shortest diameter of an ellipse.

Ex. 893. At what points of an ellipse will the normal pass through the centre of the ellipse?

Ex. 894. If FR, $F'S$ are the perpendiculars dropped from the foci to any tangent, then $FR \times F'S = b^2$.

Ex. 895. The semi-minor axis of an ellipse is the mean proportional between the segments of the major axis made by one of the foci.

Ex. 896. The area of an ellipse is to the area of its auxiliary circle as the minor axis is to the major axis.

Ex. 897. To draw a diameter conjugate to a given diameter in a given ellipse.

Ex. 898. Given $2a$, $2b$, one focus, and one point of an ellipse, to construct the ellipse.

Ex. 899. If from a point P a pair of tangents PQ and PR are drawn to an ellipse, then PQ and PR subtend equal angles at either focus.

Ex. 900. If a quadrilateral is circumscribed about an ellipse, either pair of its opposite sides subtend angles at either focus whose sum is equal to two right angles.

Ex. 901. To find the foci of an ellipse, having given the major axis and one point on the curve.

Ex. 902. To find the foci of an ellipse, having given the major axis and a straight line which touches the curve.

Ex. 903. If a straight line moves so that its extremities are always in contact with two fixed straight lines perpendicular to each other, then any point of the moving line describes an ellipse.

Ex. 904. To construct an ellipse, having given one of the foci and three tangents.

Ex. 905. To construct an ellipse, having given one focus, two tangents, and one of the points of contact.

Ex. 906. To construct an ellipse, having given one focus, one vertex, and one tangent.

Ex. 907. The area of the parallelogram formed by the tangents to an ellipse at the extremities of any pair of conjugate diameters is equal to the area of the rectangle contained by the axes of the ellipse.

Ex. 908. Given an ellipse, to find by construction the centre, the foci, and the axes.

Ex. 909. The circle described on any focal radius of an ellipse as a diameter is tangent to the auxiliary circle.

Ex. 910. If the ordinate and the tangent at any point P of an ellipse meet a diameter at H and K, respectively, then $OH \times OK = \overline{OQ}^2$. Q is the point in which the diameter cuts the curve.

THE HYPERBOLA.

945. Def. An **hyperbola** is a curve which is the locus of a point that moves in a plane so that the difference of its distances from two fixed points in the plane is constant.

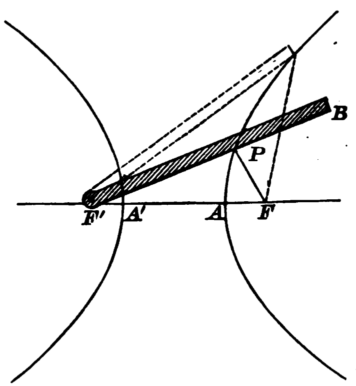
946. Def. The fixed points are called the **foci**, and the straight lines which join a point of the locus to the foci are called the **focal radii** of that point.

947. The constant difference of the focal radii is denoted by $2a$, and the distance between the foci by $2c$.

948. Def. The ratio $c:a$ is called the **eccentricity**, and is denoted by e. Therefore, $c = ae$.

949. Cor. *$2a$ must be less than $2c$* (§ 138); *hence, e must be greater than* 1.

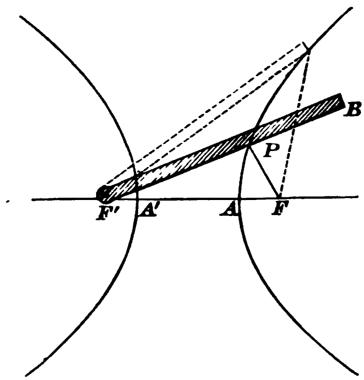
950. An hyperbola may be described by the continuous motion of a point, as follows:



To one of the foci F' fasten one end of a rigid bar $F'B$ so that it is capable of turning freely about F' as a centre in the plane of the paper.

Take a string whose length is less than that of the bar by the constant difference $2a$, and fasten one end of it at the other focus F, and the other end at the extremity B of the bar.

If now the rod is made to revolve about F' while the string is kept constantly stretched by the point of a pencil at P, in contact with the bar, the point P will trace an hyperbola.



For, as the bar revolves, $F'P$ and FP are each increasing by the same amount; namely, the length of that portion of the string which is removed from the bar between any two positions of P; hence, the difference between $F'P$ and FP will remain constantly the same.

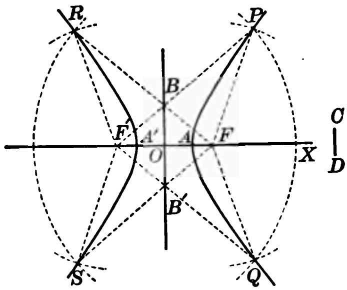
The curve obtained by turning the bar about F' is the right-hand *branch* of the hyperbola. Another similar branch on the left may be described in the same manner by making the bar revolve about F as a centre.

If the two branches of the hyperbola cut the line FF' at A and A', then, from the symmetry of the construction, $AA' = 2a$.

The hyperbola, therefore, consists of two similar branches which are separated at their nearest points by the distance $2a$, and which recede indefinitely from the line FF' and from one another.

Proposition XXV. Problem.

951. *To construct an hyperbola by points, having given the foci and the constant difference* $2a$.



Let F, F' be the foci, and CD equal a.

Lay off OA equal to OA' equal to CD.

Then A and A' are two points of the curve.

Proof. From the construction, $AA' = 2a$ and $AF = A'F'$.

Therefore, $AF' - AF = AF' - A'F' = AA' = 2a$.

And $A'F - A'F' = A'F - AF = AA' = 2a$.

To locate other points, mark any point X in $F'F$ produced. Describe arcs with F' and F as centres, and $A'X$ and AX as radii, intersecting in P, Q.

Then P and Q are points of the curve.

By describing the same arcs with the foci interchanged, two more points R and S may be found.

By assuming other points in $F'F$ produced, any number of points may be found; and the curve passing through all these points is an hyperbola having F, F' for foci and $2a$ for the constant difference of the focal radii. Q.E.F.

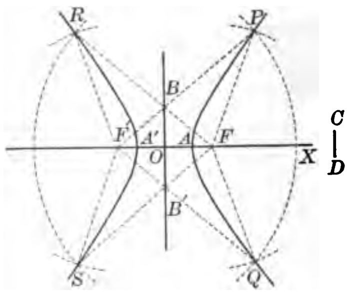
952. Cor. 1. *No point of the curve can be situated on the perpendicular to* FF' *erected at* O.

For every point of this $\perp$ is equidistant from the foci.

953. Def. The point O is called the **centre**; AA' is called the **transverse axis**; A and A' are called the **vertices.**

954. Def. In the $\perp$ to FF' erected at O, let B, B' be two points at a distance from A (or A') equal to c; then BB' is called the **conjugate axis,** and is denoted by $2b$.

955. Def. If the transverse and conjugate axes are equal, the hyperbola is said to be **equilateral** or **rectangular.**



956. Cor. 2. *Both the axes are bisected at the centre.*

957. Cor. 3. *By* § 371, $c^2 = a^2 + b^2$.

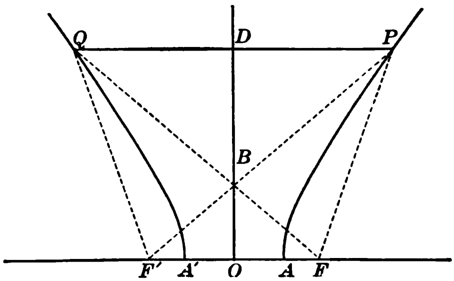
958. Cor. 4. *The curve is symmetrical with respect to the transverse axis.*

959. Def. The distances of a point of the curve from the transverse axis and the conjugate axis are called respectively the **ordinate** and **abscissa** of the point. The double ordinate through the focus is called the **latus rectum** or **parameter.**

Note. The letters A, A', B, B', F, F', and O will be used to designate the same points as in the above figure.

PROPOSITION XXVI. THEOREM.

960. *An hyperbola is symmetrical with respect to its conjugate axis.*



Let P be a point of the curve, PDQ be perpendicular to OB, meeting OB at D, and let DQ equal DP.

To prove that Q is also a point of the curve.

Proof. Join P and Q to the foci F, F'.

Turn $ODQF'$ about OD; F' will fall on F, and Q on P.

Therefore, $QF' = PF$, and $\angle PQF' = \angle QPF$.

Therefore,	$\triangle PQF' = \triangle QPF$,	§ 143
and	$QF = PF'$.	§ 128
Hence,	$QF - QF' = PF' - PF$.	Ax. 3
But	$PF' - PF = 2a$.	Hyp.
Therefore,	$QF - QF' = 2a$.	Ax. 1
Therefore,	Q is a point of the curve.	§ 945

Q. E. D.

961. DEF. Every chord passing through the centre is called a **diameter.**

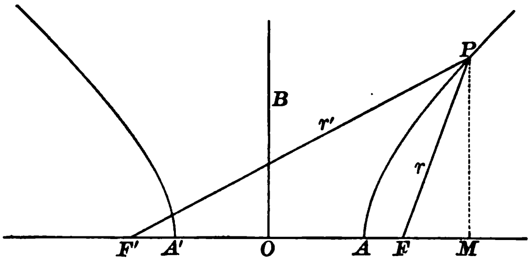
962. COR. 1. *An hyperbola consists of four equal quadrantal arcs symmetrically placed about the centre.* § 213

963. COR. 2. *Every diameter is bisected at the centre.* § 209

Proposition XXVII. Theorem.

964. *If d denotes the abscissa of a point of an hyperbola, r and r' its focal radii, then*

$$r = ed - a, \text{ and } r' = ed + a.$$



Let P be any point of the hyperbola, PM perpendicular to AA', d equal OM, r equal PF, r' equal PF'.

To prove that $r = ed - a,\ r' = ed + a.$

Proof. From the rt. $\triangle$ FPM, $F'PM$,

$$r^2 = \overline{PM}^2 + \overline{FM}^2, \qquad \text{§ 371}$$

$$r'^2 = \overline{PM}^2 + \overline{F'M}^2.$$

Therefore, $r'^2 - r^2 = \overline{F'M}^2 - \overline{FM}^2.$

Or $(r' + r)(r' - r) = (F'M + FM)(F'M - FM).$

Now $r' - r = 2a$, and $F'M - FM = 2c.$

Also $F'M + FM = 2\,OF + 2\,FM = 2\,OM = 2d.$

By substitution, $a(r' + r) = 2cd.$

Or $$r' + r = \frac{2cd}{a} = 2ed.$$

From $r' + r = 2ed$, and $r' - r = 2a$,

by addition, $2r' = 2(ed + a)$;

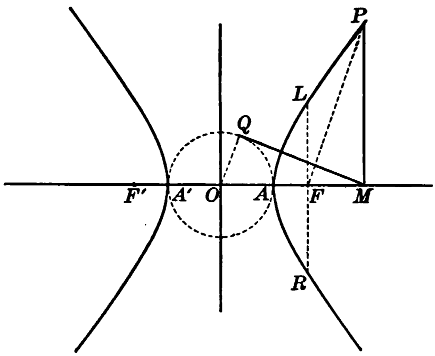
by subtraction, $2r = 2(ed - a)$.

Therefore, $r = ed - a$, and $r' = ed + a$. Q.E.D.

965. Def. The circle described upon AA' as a diameter is called the **auxiliary circle.**

Proposition XXVIII. Theorem.

966. *Any ordinate of an hyperbola is to the tangent from its foot to the auxiliary circle as b is to a.*



Let P be any point of the hyperbola, PM the ordinate, MQ the tangent drawn from M to the auxiliary circle.

To prove that $PM : QM = b : a.$

Proof. Let OM equal d.

Then $\overline{QM}^2 = d^2 - a^2.$ § 371

Also $\overline{PM}^2 = \overline{PF}^2 - \overline{FM}^2$

$= (ed - a)^2 - (d - c)^2$ § 964

$= e^2d^2 - 2\,aed + a^2 - d^2 + 2\,cd - c^2.$

Or since $c = ae$, and $a^2 - c^2 = -b^2$, §§ 948, 957

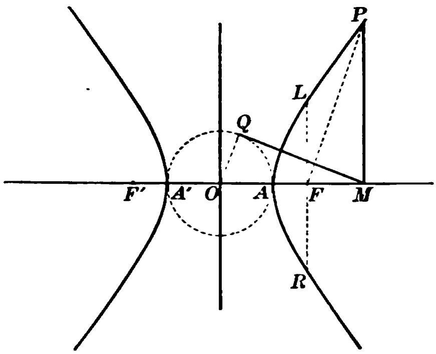
$$\overline{PM}^2 = (e^2 - 1)\,d^2 - b^2 = \frac{b^2}{a^2}(d^2 - a^2).$$

Therefore, $\overline{PM}^2 : \overline{QM}^2 = b^2 : a^2.$

Or $PM : QM = b : a.$ Q. E. D.

PROPOSITION XXIX. THEOREM.

967. *The square of the ordinate of a point in an hyperbola is to the product of the distances from the foot of the ordinate to the vertices as b^2 is to a^2.*



Let P be any point of the hyperbola, PM the ordinate, MQ the tangent from M to the auxiliary circle.

To prove that $\overline{PM}^2 : AM \times A'M = b^2 : a^2$.

Proof. Now $\overline{PM}^2 : \overline{QM}^2 = b^2 : a^2$. § 966

But $\overline{QM}^2 = AM \times A'M$. § 381

Therefore, $\overline{PM}^2 : AM \times A'M = b^2 : a^2$. Q. E. D.

968. COR. *The latus rectum is the third proportional to the transverse and conjugate axes.*

For $\overline{LF}^2 : AF \times A'F = b^2 : a^2$. § 967

But $AF = c - a$, and $AF' = c + a$.

Therefore, $AF \times A'F = c^2 - a^2 = b^2$. § 957

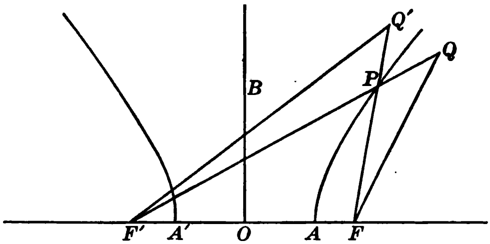
Hence, $\overline{LF}^2 : b^2 = b^2 : a^2$.

And $LF : b = b : a$.

Therefore, $2a : 2b = 2b : 2LF$.

PROPOSITION XXX. THEOREM.

969. *The difference of the distances of any point from the foci of an hyperbola is greater than or less than* $2a$, *according as the point is on the concave or convex side of the curve.*



1. Let Q be a point on the concave side of the curve.

To prove that $QF' - QF > 2a$.

Proof. Let QF' meet the curve at P.

$F'Q = F'P + PQ$ (Ax. 9), and $FQ < FP + PQ$. § 138

$\therefore F'Q - FQ > F'P - FP$. Ax. 5

But $F'P - FP = 2a$. § 947

Therefore, $F'Q - FQ > 2a$.

2. Let Q′ be a point on the convex side of the curve.

To prove that $Q'F' - Q'F < 2a$.

Proof. Let $Q'F$ cut the curve at P.

$F'Q' < F'P + PQ'$ (§ 138), and $FQ' = FP + PQ'$. Ax. 9

$\therefore F'Q' - FQ' < F'P - FP$. Ax. 5

But $F'P - FP = 2a$. § 947

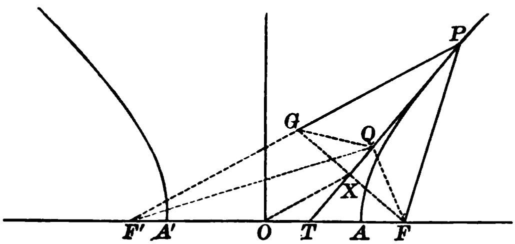
Therefore, $F'Q' - FQ' < 2a$. Q.E.D.

970. COR. *Conversely, a point is on the concave or the convex side of the hyperbola according as the difference of its distances from the foci is greater than or less than* $2a$.

971. Def. A straight line which touches but does not cut the hyperbola is called **a tangent**, and the point where it touches the hyperbola is called the **point of contact to the hyperbola.**

Proposition XXXI. Theorem.

972. *If through a point* P *of an hyperbola a line is drawn bisecting the angle between the focal radii, every point in this line except* P *is on the convex side of the curve.*



Let PT bisect the angle FPF', and let Q be any point in PT except P.

To prove that Q is on the convex side of the curve.

Proof. Take PG equal to PF; draw FG, QF, QF', QG.

Then $\qquad QF' - QG < GF'.$ § 138

Also $\triangle PGQ = \triangle PFQ$ (§ 143); $\therefore QG = QF.$

Also $\qquad GF' = PF' - PF = 2a.$

Therefore, $\qquad QF' - QF < 2a.$

Therefore, Q is on the convex side of the curve. § 970

Q. E. D.

973. Cor. 1. *The bisector of the angle between the focal radii from any point* P *is the tangent to the curve at* P. § 971

974. Cor. 2. *The tangent to an hyperbola at any point bisects the angle between the focal radii drawn to that point.*

975. Cor. 3. *The tangent at* A *is perpendicular to* AA'.

976. Cor. 4. *If* FG *cuts* PT *at* X, *then* $GX = FX$, *and* PT *is perpendicular to* FG. § 161

977. Cor. 5. *The locus of the foot of the perpendicular from the focus to a tangent is the auxiliary circle.*

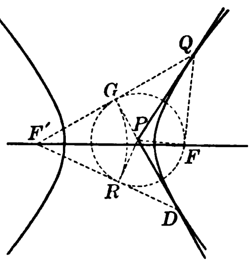
For $FX = GX$, and $FO = OF'$.

$\therefore OX = \frac{1}{2} F'G = \frac{1}{2}(PF' - PF) = a.$ § 189

Therefore, the point X lies on the auxiliary circle.

Proposition XXXII. Problem.

978. *To draw a tangent to an hyperbola from a given point* P *on the convex side of the hyperbola.*



Let the arcs described with P as centre and PF as radius, and with F' as centre and $2a$ as radius intersect in G and R.

Draw $F'G$ and $F'R$, and produce them to meet the curve in Q and D, respectively. Draw PQ and PD.

PQ and PD are the tangents required.

Proof. $PG = PF$, $QF = QF' - 2a = QG$.

$\therefore \triangle PQG = \triangle PQF.$ § 150

$\therefore \angle PQG = \angle PQF.$ § 128

$\therefore PQ$ is the tangent at Q. § 973

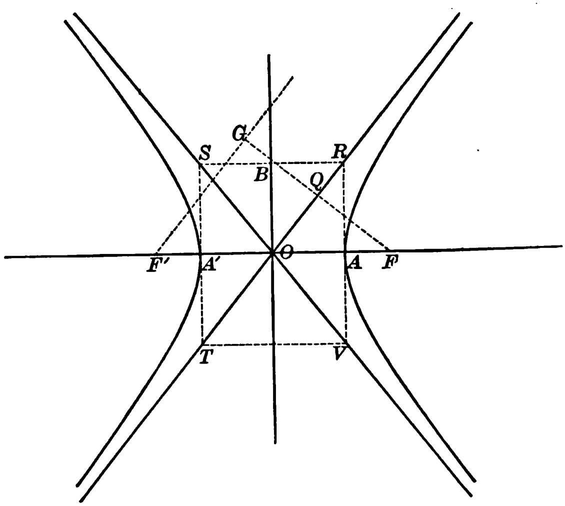
For like reason, PD is the tangent at D. Q. E. F.

979. Cor. *Two tangents may be drawn to an hyperbola from a point on the convex side of the hyperbola.*

980. Def. If a rectangle is constructed with its adjacent sides equal, respectively, to the transverse and conjugate axes of the hyperbola, and with one side tangent to the curve at A and its opposite side at A', its diagonals produced are called the **asymptotes of the hyperbola.**

Proposition XXXIII. Theorem.

981. *The asymptotes of an hyperbola never meet the curve, however far produced.*



Let TR be an asymptote of the hyperbola whose centre is O.

To prove that TR never meets the curve.

Proof. Let G be the intersection of arcs described from O and F' as centres with OF and $2a$, respectively, as radii.

If TR meets the curve, the point of contact must be at the intersection of $F'G$ and TR. § 978

Draw FG, cutting TR at Q.

Now $OF' = OF$;

also $QG = QF$. § 976

$\therefore F'G$ is $\parallel$ to TR. § 189

Therefore, $F'G$ and TR cannot intersect. § 103

Therefore, TR does not meet the curve. Q. E. D.

982. Cor. 1. *The line* FG *is tangent to the auxiliary circle at* Q.

For FG is $\perp$ to OR. § 976

Therefore, Q lies on the auxiliary circle. § 977

Hence, FG touches the auxiliary circle at Q. § 253

983. Cor. 2. FQ *is equal to the semi-conjugate axis* b.

For $\overline{FQ}^2 = \overline{OF}^2 - \overline{OQ}^2$, § 372

and $b^2 = c^2 - a^2$. § 957

But $OF = c$, and $OQ = a$.

Therefore, $FQ = b$.

984. Cor. 3. *If the tangent to the curve at* A *meets the asymptote* OR *at* R, *then* $AR = b$.

For $\triangle\, OAR = \triangle\, OQF$. § 142

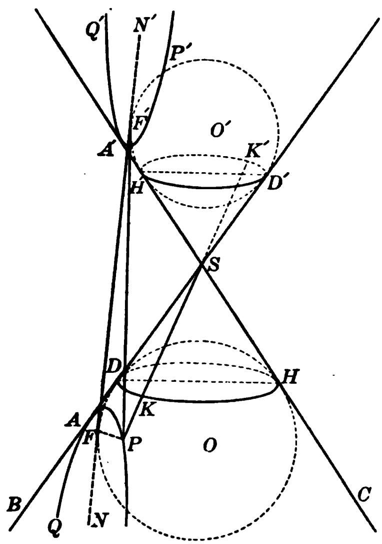
Therefore, $AR = FQ = b$.

985. Def. A perpendicular to a tangent erected at the point of contact is called a **normal**.

986. Def. The terms **subtangent** and **subnormal** are used in the hyperbola in the same sense as in the ellipse. § 934

Proposition XXXIV. Theorem.

987. *The section of a right circular cone made by a plane that cuts both nappes of the cone is an hyperbola.*



Let a plane cut the lower nappe of the cone in the curve PAQ, and the upper nappe in the curve P′A′Q′.

To prove that PAQ and $P'A'Q'$ are the two branches of an hyperbola.

Proof. The plane passed through the axis of the cone perpendicular to the secant plane cuts the surface of the cone in the elements BS, CS (prolonged through S), and the secant plane in the line NN'.

Describe the Ⓢ O, O', tangent to BS, CS, NN'. Let the points of contact be D, H, F, and D', H', F', respectively.

Turn BSC and the ⓢ O and O' about the axis of the cone. BS and CS will generate the surfaces of the two nappes of a right circular cone; and the ⓢ O, O' will generate spheres which touch the cone in the ⓢ DKH, $D'K'H'$, and the secant plane in the points F, F'.

Let P be any point on the curve. Draw PF and PF'; and draw PS, which touches the ⓢ DKH, $D'K'H'$, at the points K, K'.

Now PF and PK are tangents to the sphere O from the point P.

Therefore, $PF = PK$. § 261

Also $PF' = PK'$. § 716

Hence, $PF' - PF = PK' - PK$ Ax. 3

$= KK'$, a constant quantity. § 716

Therefore, the curve is an hyperbola with the points F and F' for foci. § 945

Q. E. D.

TABLE OF FORMULAS.

SOLID GEOMETRY.

S = lateral area. V = volume.

E = lateral edge; element. H = altitude.

P = perimeter of right section (Prisms and Cylinders).

p = perimeter of upper base. L = slant height.

P = perimeter of lower base. B = lower base.

c = circumference of upper base. b = upper base.

C = circumference of lower base. T = total area.

r = radius of upper base. M = area of mid-section.

R = radius of lower base.

a, b, c = dimensions of parallelopiped.

FORMULAS.

Prisms and Parallelopipeds.

	PAGE
$S = E \times P$	293
$V = B \times H$	303, 305
$V = a \times b \times c$ (Rectangular Parallelopiped)	302

Pyramids.

$S = \frac{1}{2} L \times P$ (Regular Pyramid)	309
$V = \frac{1}{3} B \times H$	316

Frustums of Pyramids.

$S = \frac{1}{2}(P + p)L$ (Regular Pyramid)	309
$V = \frac{1}{3} H(B + b + \sqrt{B \times b})$ (Any Pyramid)	320

Cylinders of Revolution.

	PAGE
$S = C \times H = 2\pi RH$	338, 339
$T = 2\pi R(H + R)$	339
$V = B \times H$ (Any Cylinder)	339
$V = \pi R^2 H$	339

Cones of Revolution.

$S = \frac{1}{2} C \times L = \pi RL$	348
$T = \pi R(L + R)$	348
$V = \frac{1}{3} B \times H$ (Any Cone)	349
$V = \frac{1}{3}\pi R^2 H$	349

Frustums of Cones of Revolution.

$S = \frac{1}{2}(C + c) \times L$	351
$V = \frac{1}{3} H(B + b + \sqrt{B \times b})$ (Any Frustum)	352
$V = \frac{1}{3}\pi H(R^2 + r^2 + R \times r)$	353

Prismatoids.

$V = \frac{1}{6} H(B + b + 4M)$	354

R = radius of sphere. D = diameter of sphere.

S = area of surface. H = altitude.

A = number of degrees in angle. B = base.

E = spherical excess. r, r' = radii of bases.

V = volume. T = sum of angles.

n = number of sides.

Spherical Areas.

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		PAGE
Lune,	$S = \dfrac{\pi R^2 A}{90}$	393
Spherical triangle,	$S = \dfrac{\pi R^2 E}{180}$	395
Spherical polygon,	$E = T - (n - 2)180$ . . .	396
	$S = \dfrac{\pi R^2 E}{180}$	396

Spherical Volumes.

Sphere,	$V = \frac{4}{3}\pi R^3$. $V = \frac{1}{6}\pi D^3$ . .	399
Spherical pyramid,	$V = \frac{1}{3} R \times B$	399
Spherical sector,	$V = \frac{2}{3}\pi R^2 H$	400
Spherical segment,	$V = \frac{1}{2} H(\pi r^2 + \pi r'^2) + \frac{1}{6}\pi H^3$.	401

Ellipse.

	$S = \pi ab$	441

INDEX OF DEFINITIONS.

List of pages on which are corrections to be made at the press - I shall speak of ~~a~~ when going to the press. The changes referred to on ~~the~~ opposite page are for your consideration. M.G.C.

P. 260 - several changes
264
269 - two changes
276
283 two changes
295
298
300 - one change -
312 two changes
318
321
324
328
[illegible]
356
399

List of Changes

(Insertions — changes of text etc
addition of lines etc.)

P-256 - Draw BD dotted -

267 - Insert commas - see p.

279 - On Friday morning you
spoke I believe of changing
as shown on 279 -

308 - necessity of lettering diagram questioned

320 - on Friday you suggested
change - see page 320

331 - See page -

332 necessity of lettering questioned

333 - ~~Fact~~ Reference inserted see p. 333

343 - Word inserted see p. 343
need of letter questioned

346 - addition of line a-o-c - } given
A-O-C - } Miss Wh
Fri.

347 - trans ... te

392 impor ...